ECON 8101-8104- MINI I-IV

Prelims (updated 04/09/21) Microeconomics

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1 Werner

Question I.1 Fall 2004 majors

Let $u : \mathbb{R}_{+}^{L} \rightarrow \mathbb{R}$ be a continuous and locally non-satiated utility function. For a vector of prices of L goods $p \in \mathbb{R}_{++}^{L}$ and income $M > 0$, let $u^{*}(p, M)$ denote the indirect utility function associated with the Walrasian demand for u

(a) Show that function u^{*} is quasi-convex in p

(b) Suppose that u is quasi-linear of the form

$$u\left(x_{1}, x_{2}, \ldots, x_{L}\right)=x_{1}+v\left(x_{2}, \ldots, x_{L}\right)$$

for some strictly increasing and strictly concave function $v : \mathbb{R}_{+}^{L-1} \rightarrow \mathbb{R}$.
Show that indirect utility u^{*} is a linear function of income M on the domain of price- income pairs (p, M) where the Walrasian demand is interior. If your proof relies on differentiability of the demand function and/or the indirect utility function, you should clearly state additional assumptions pertaining to differentiability of the function v that are sufficient for these desired properties of demand and indirect utility, and justify sufficiency of your assumptions.

2 Rustichini

Question I.2 Fall 2004 majors

Let X be a finite set of outcomes and $\mathcal{L}$ the set of all lotteries on X. Further, let $\succeq$ be a (complete and transitive) preference relation on $\mathcal{L}$.

(a) State the independence axiom for $\succeq$.

(b) Give an example of preference relation $\succeq$ that does not satisfy the independence axiom.

(c) Show that, if $\succeq$ is contimous and satisfies the independence axiom, then for all lotteries L_1, L_2, L such that $L_1 \succ L \succ L_2$, there exists unique $\alpha \in [0, 1]$ such that $L \sim L_1 \alpha L_2$, where $L_1 \alpha L_2$ denotes the compound lottery.

3 Allen

Question II.1 Fall 2004 majors

This question concerns the three main results (and their proofs) in general equi- librium theory - existence of competitive equilibrium, the first welfare theorem, and the second welfare theorem - in pure exchange economies with ℓ commodities (the commodity space is $\mathbb{R}^{\ell}$) and N consumers, each having consumption set Π^{ℓ}_{+}, initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$, and preferences $\preceq_i$ (for $i = 1, \ldots, N$) which are assumed throughout to be strictly monotone continuous complete preorders on $\mathbb{R}^{\ell}_{+}$.

(i) Define competitive equilibrium in such economies.

(ii) Define the set of Pareto optimal allocations in such economies. Now consider three different versions of these economies.

(a) In economy (a), preferences $\preceq_i$ (for $i = 1, \ldots, N$) are assumed to be strictly (or strongly) convex.

(b) In economy (b), preferences $\preceq_i$ (for $I = 1, \ldots, N$) are assumed to be convex.

(c) In economy (c), preferences $\preceq i$ (for $i = 1, \ldots, N$) are not necessarily even weakly convex.

(iii) Does existence of competitive equilibrium hold in economy (a)? In econ- omy (b)? In economy (c)? For your first "no" answer (if any), explain why briefly. For your second and third "yes" answers (if any), explain how and why the proof of the theorem becomes more difficult/complicated com- pared to the previous "yes" case (if it does) and what method(s) can be used to solve the difficulties.

(iv) Does the first welfare theorem hold in economy (a)? In economy (b)? In economy (c)? For your first "no" answer (if any), explain why briefly. For your second and third "yes" answers (if any), explain how and why the proof of the theorem becomes more difficult/complicated compared to the previous "yes" case (if it does) and what method(s) can be used to solve the difficulties.

(v) Does the second welfare theorem hold in economy (a)? In economy (b)? In economy (c)? For your first "no" answer (if any), explain why briefly. For your second and third "yes" answers (if any), explain how and why the proof of the theorem becomes more difficult/complicated compared to the previous "yes" case (if it does) and what method(s) can be used to solve the difficulties. [Implicitly, you are asked to fill in a "true" or "false" 3 ×3 matrix (three theorems in economies (a), (b), and (c)) and then to explain any answer of "false" as well as answers of "true" for economy (b) or (c).]

4 Allen

Question II.2 Fall 2004 majors

Consider a pure exchange economy with ℓ commodities (The commodity space is $\mathbb{R}^\ell$) and N traders (denoted $i = 1, \ldots, N$), each having consumption set $\mathbb{R}^\ell_+$, init ial endowment vector $e_i \in \Pi^\ell_{++}$, and preferences $\preceq_i$ (for $i = 1, \ldots, N$) which are assumed to be continuous complete preorders on $\mathbb{R}^\ell_+$. Suppose that all commodities are "bads" in the sense that for any $x, y \in \mathbb{R}^\ell_+$ with $x \geqq y$ but $x \neq y$, it is true that $y \succ_i x$ for all $i = 1, \ldots, N$ (a) Can these preferences $\preceq_i$ be represented by a utility function $u_i : \mathbb{R}^\ell_+ \to \mathbb{R}$? Sketch a proof (if yes) or indicate why there is no utility representation (if no).

(b) State the maximum theorem. Can it be used to obtain properties of demand in this economy?

(c) Assuming free disposal, find all competitive equilibria in this economy.

(d) Assuming free disposal, find all Pareto optimal allocations in this economy.

(e) Still assuming free disposal, does the first welfare theorem apply in this economy?Why or why not?

(f) Still assuming free disposal, does the second welfare theorem apply in this economy? Why or why not?

(g) Now suppose that there is no free disposal (and no disposal technology). In answering part (g), you should add a convexity assumption and precisely state it. Discuss the possibilities for existence of competitive equilibrium in this econ- omy. In particular, define competitive equilibrium, discuss your choice of the price space, and either sketch a proof of existence or give a counterexample to the existence of competitive equilibrium even if prices are permitted to be negative or zero.

5 Rustichini

Question III.1 Fall 2004 majors

(a) For iterated elimination of strictly dominated strategies, show that the sets are nested and that the procedure terminates in finitely many rounds if the game is finite.

(b) Show that any strategy that survives iterated weak dominance also survives iterated strict dominance.

(c) State and prove a tight upper bound on the number of iterations that may be required.

6 Rahman

Question III.2 Fall 2004 majors

Consider an auction in setting in which n bidders are competing for an object. The agent's valuations satisfy the independent private values model with each agent's valuation distributed on the interval [0, 1] according to the cumulative distribution function $F(v) = v^3$

(a) Prove that there is a unique Bayes Nash equilibrium of the second price sealed bid auction.

(b) Find the symmetric monotone equilibrium of the first price sealed bid auction.

(c) Compute the expected of the equilibrium given in (b).

7 Allen

Question IV.1 Fall 2004 majors

Consider an economy with two consumer-workers and a firm producing a public good or service Y using the workers' labor X. There are two goods: Y, and a private good Z (leisure), of which each consumer has a positive endowment. There are no en- dowments of Y. Labor X is the sacrifice of leisure.
The firm's technology is represented by a convex strictly increasing input requirements function $x = g(y)$, with $g(0) = 0$. The firm is a profit maximizer and is owned by consumer 1. There is no free disposal.

ANSWER (a) AND EITHER (b) OR (c).

(a) Define a Lindahl equilibrium for this economy. (Write down explicitly the feasibility conditions applicable to this economy, including the appropriate balance relation.)

(b) Prove that Lindahl equilibrium allocations are (Strongly) Pareto optimal when constant returns to scale prevail in the productions of Y. State explicitly all your assumptions and show where and how they are used.

(c) Prove that Lindahl equilibrium allocations are Weakly Pareto optimal without assuming constant returns to scale. State explicitly all your assumptions and show where and how they are used.

8 Allen

Question IV.2 Fall 2004 majors

Consider a pure exchange economy without free disposal, with two consumers (1, 2) and two goods (X, Y). Each consumer has an initial endowment of three units of each good. Each consumer's consumption set is the nonnegative quadrant, i.e., $C^i = \mathbb{R}^2_+$ $i = 1, 2$. Preferences are represented by utility functions $U_i = f_i(x_i, y_i, x_j, y_j)$, where $j \neq i$. That is, each consumer may care not only about his/her own consumption but possibly also about the other person's consumption.

(a.i.) Define a (Strongly) Pareto optimal allocation in this economy.
(a.ii.) Define a Walrasian equilibrium in this economy.

(b) Suppose the utility functions are given as $u_1 = x_1 + y_1 + 3x_2$, and $u_2 = x_2 + y_2$

(b.i.) What are the Walrasian equilibrium allocations in this economy? (Specify algebraically and show clearly in an Edgeworth Box diagram.) What are the corresponding equilibrium prices of the two goods?
(b.ii.) Are these Walrasian equilibrium allocations (Strongly) Pareto optimal? Give a complete and rigorous justification for your answer.

9 Werner

Question I.1 Spring 2005 majors

Consider the following optimal insurance problem: An agent with preferences rep- resented by an expected utility function has a deterministic initial wealth $w > 0$ and faces a risk of losing L such that $0 < L < w$. The agent can purchase insurance at a premium $\rho \in (0, 1)$ per one dollar of coverage to be paid in case of loss. The probability of loss is $\pi \in (0, 1)$

The agent's optimal-insurance problem is

$$\max_{a \geq 0} \pi u(w - L + a - \rho a) + (1 - \pi)u(w - \rho a)$$

The agent is strictly risk averse. The agent's utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and twice-differentiable. Suppose that

(i) Show that the optimal insurance is less than full insurance.

(ii) Show that the agent's demand for insurance is an increasing function of the amount of loss L

10 Werner

Question I.2 Spring 2005 majors

Let $\succeq$ be a reflexive, transitive and complete preference relation on $\mathbb{R}^L_+$.

(a) Prove that if $\succeq$ is strictly convex and increasing (i.e., $x \geq x'$ implies that $x \succeq x'$, for every $x, x' \in \mathbb{R}^L_+$), then it is locally non-satiated.

(b) Prove that if $\succeq$ is locally non-satiated, then every consumption bundle x in the Walrasian demand $x^*(p, M)$ must satisfy the budget equation $px = M$, for every vector of prices $p \in \mathbb{R}^L_{++}$ and income $M > 0$.
Write your proof so that it clearly indicates how your definition of local non-satiation is used. A graphical illustration is not enough.

(c) Is local non-satiation also a necessary condition for every consumption bundle in the Walrasian demand to satisfy the budget constraint with equality, for every $p \in R^L_{++}$ and $M > 0$? Either prove that it is, or give a counterexample. You may use a graphical illustration to support an ex- ample.

11 Allen

Question II.1 Spring 2005 majors

It is frequently observed that microeconomic theory is based on constrained optimization. This question asks you to explore the consequences of departing from this paradigm, specifically by considering the effect of "satisficing" behavior by traders. As usual, we consider a perfectly-competitive pure exchange economy with n (finite integer) consumers and ℓ (finite integer) commodities. Each consumer $i = 1, \ldots, n$ has consumption set $\mathbb{R}^{\ell}_{++}$, initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$, and utility function $u_i : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$ which is assumed to be twice continuously differentiable, strictly monotone, and strictly concave. Each utility u_i is also assumed to satisfy the boundary condition that the closure in $\mathbb{R}^{\ell}$ of each set $\left\{x \in \mathbb{R}^{\ell}_{++} | u_i(x) \geq u_i(\bar{x})\right\}$ is disjoint from the boundary of $\mathbb{R}^{\ell}_{+}$ (for all $\bar{x} \in \mathbb{R}^{\ell}_{++}$ and all $i = 1, \ldots, n$). For $i = 1, 2, \ldots, n$, arbitrarily fix $\bar{u}_i \in \mathbb{R}$. Satisficing will be defined to mean that the usual individual demand correspondences are replaced by the correspondences $\hat{X}_i : \Delta \to \mathbb{R}^{\ell}_{++}$ (for $i = 1, \ldots, n$, where $\Delta = \{p \in \mathbb{R}^{\ell}_{++} | \sum_{k=1}^{\ell} p_k = 1\}$ denotes the relatively open $(\ell - 1)$ -dimensional unit price simplex in $\bar{R}^{\ell}$) defined by $\hat{X}_i(p) = \left\{x \in \mathbb{R}^{\ell}_{++} | u_i(x) \geq \bar{u}_i \text{ and } p \cdot x \leq p \cdot e_i\right\}$. In answering the following questions, be sure to justify or explain your answers precisely.

(a) State the Maximum Theorem. What (if anything) does it imply about the properties of the correspondence $\hat{X}_i : \Delta \to \mathbb{R}^{\ell}_{++}$?

(b) Define competitive equilibrium for this model. Does competitive equilibrium exist in this model? If not, are there additional assumptions (state them) on the vector $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ that guarantee the existence of competitive equilibrium in this model? Explain; prove your assertion.

(c) State the first welfare theorem. What does it imply about the relationship between competitive equilibria and Pareto optimality in this model?

(d) Now discuss the properties of smooth economies in this model. In particular, what can be said about the potential finiteness of the equilibrium price set?

(e) Finally, replace the $\hat{X}_i$ correspondences by the correspondences $\tilde{X}_i : \Delta \to \mathbb{R}^{\ell}_{++}$ defined by $\tilde{X}_i(p) = \left\{x \in \mathbb{R}^{\ell}_{++} | u_i(x) > \bar{u}_i \text{ and } p \cdot x \leq p \cdot e_i\right\}$. Would your answers to (a), (b), (c), and (d) above change? If so, how (be precise)? Explain/justify your answers.

12 Allen

Question II.2 Spring 2005 majors

It is sometimes claimed that the presence of nonconvexities creates problems for general equilibrium theory because they cause failures of the continuity needed to use Kakutani's fixed point theorem. This question refers to a perfectly-competitive pure ex- change economy with ℓ commodities and n consumers, each having initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$ and preferences $\leq_i$ which are strictly monotone complete continuous preorders on $\mathbb{R}^{\ell}_{+}$

(a) State Kakutani's fixed point theorem and briefly explain how it is used to establish existence of competitive equilibrium.

(b) Suppose that each trader's consumption set is $\mathbb{R}^{\ell}_{+}$ but preferences $\leq_i$ can fail tobe even weakly convex. Is it true that this causes continuity failures which render it impossible to apply Kakutani's fixed point theorem (to obtain existence of competitive equilibrium)? Explain your answer.

(c) Now suppose that each consumer's preferences $\leq i$ are convex, but each consumer is restricted to his or her consumption set X_i, which is assumed to be a nonempty closed subset of $\mathbb{R}^{\ell}_{+}$. Is it true that this causes continuity failures which render it impossible to apply Kakutani's fixed point theorem (to obtain existence of competitive equilibrium)? Explain your answer.

(d) State the first welfare theorem. Does it apply to economies with nonconvex pref- erences (as in part (b))? Does it apply to economies with convex preferences but consumption sets X_i as in part (c)? For each question, if your answer is yes, give a proof and if your answer is no, provide a clear counterexample.

13 Rustichini

Question III.1 Spring 2005 majors

Let G be a normal form game,

$$G \equiv \left(I, \left(A^i, u^i\right)_{i\in I}\right)$$

where I is the set of players, and for every i, A^i is the set of pure strategies and u^i the utility function of i. Let also G' be a game obtained from G by eliminating pure strate- gies that are weakly dominated in G.
Let σ' be a mixed strategy Nash equilibrium of G', and let σ be its natural ex- tension to a strategy for G, obtained by assigning zero probability to the pure strategies that have been eliminated in G'

(a) Prove that σ is a Nash equilibrium of G.

(b) Consider now the same question, but let the extension σ assign zero probability to the pure strategies that have been eliminated in an iterated elimination of weakly dominated strategies. Is the extension σ a Nash equilibrium? Prove your answer.

14 Rustichini

Question III.2 Spring 2005 majors

Let $N = (S_1, \ldots, S_n; u_1, \ldots, u_n)$ be a normal form game.

(a) Define the notion of a perfect equilibrium for N.

(b) Find all the Nash equilibria and all the perfect equilibria of the normal form game shown below.

$$\begin{pmatrix} & L & M & R \\ U & (2,2) & (1,1) & (1,1) \\ C & (1,1) & (1,1) & (0,1) \\ D & (0,2) & (1,3) & (0,0) \end{pmatrix}$$

15 Rustichini

Question IV.1 Spring 2005 majors

This question deals with public services supported by voluntary contributions of labor. The amount contributed by agent i, denoted by t_i is interpreted as i' s strategy in a non-cooperative game.

There are two goods: leisure Z (a private or rivalrous good) and a public service Y (public or non-rivalrous). Each agent i has the utility function $u^i = z_i + c_i \ln(y+1)$ $i = 1,2; c_i > 0$, and the non-negative quadrant is its consumption set, $i = 1,2$. (z_i is the amount of leisure left to i after the contribution of t_i, which is assumed nonnegative and y is the quantity of the public service.) Each agent has a positive endowment ω_i of leisure but no endowment of Y. The service Y is produced with labor (i.e., contributions of leisure, $t_i = \omega_i - z_i, i = 1,2$) as input. The production function is linear, with one unit of labor producing one unit of the service $(y = t_1 + t_2)$

In the non-cooperative game, i 's payoff function $\pi^i(t_1, t_2)$ is obtained by substituting into his/her utility function $\omega_i - t_i$ for z_i, and $t_1 + t_2$ for y

We are interested in the interior Nash equilibria. ("Interior" here means $y > 0$, $z_i > 0, i = 1,2$; the t_i 's are nonnegative but t_i zero values are not ruled out.) The inte- rior Nash equilibrium values of the variables are marked by asterisks. Assume throughout $c_1 > c_2 > 0$ and $y^* > 0$

(a) In the interior Nash equilibrium, will both agents be contributing (i.e., $t_1^* > 0$ $t_2^* > 0$) or only one (i.e., $t_i^* > 0$ for some $i, t_j^* = 0$ for $j \neq i$)? If your answer is "only one," which one will be the free rider? Give a proof of your claim. If your answer is "both," give a proof.

(b) Will the interior Nash equilibrium allocation be Pareto Optimal? If not, how does the Nash level y^* compare with the optimal level $\hat{y}$?

16 Allen

Question IV.2 Spring 2005 majors

In an exchange economy without free disposal there are two agents: the mother (agent 1) and the child (agent 2), and two goods X and Y. Each consumption set is $C^i = IR^2_+, i = 1,2$; the non-negative quadrant is the space of each agent's bundles (x_i, y_i) consumed by agent i. All endowments are equal and positive (i.e., $\omega_x^1 = \omega_y^1 = \omega_x^2 = \omega_y^2 > 0$) Their respective utility functions are

$$\begin{array}{ll} u^1 = x_1 + y_1 + kx_2 & (k > 1) \\ u^2 = & x_2 + y_2 \end{array}$$

(a.i) Define a (Strongly) Pareto-Optimal allocation, appropriate for this type of economy.
(a.ii) Formulate a (generalized) concept of Walrasian equilibrium, appropriate for this type of economy.

(b) Remembering that $k > 1$, are the Walrasian equilibrium allocations in this economy Pareto-Optimal? Give a complete justification for your answer.

17 Werner

Question I.1 Fall 2005 majors

Let X and Z be two real-valued random variables on some probability spac $(\Omega, \mathcal{F}, P)$ Suppose that $E(Z) = 0$

(a) Consider the following statement: for every X and Z with $E(Z) = 0, X + Z$ is more risky than X. Show that this statement is false

(b) Under what additional condition(s) on Z and/or X is the statement from part (a) true. Your conditions should not restrict X to be deterministic Be as general as you can. Give an example of non-deterministic X and Z that satisfy your condi- tions

(c) Prove your statement from part (b).

18 Werner

Question I.2 Fall 2005 majors

Consider a reflexive, transitive, and complete preference relation on $\mathbb{R}^n_+$. For every consumption bundle $x \in \mathbb{R}^n_+$, let $u(x) \in \mathbb{R}_+$ be a number (if it exists) such that x is indif- ferent to $u(x)$ e, where e denotes the unit vector in $\mathbb{R}^n_+$.

(a) Show that, if the preference relation is continuous and strictly increasing (also called strongly monotone), then u is a well-defined function on R^n_+ and constitutes a utility representation of the preference relation. Clearly indicate where in your proof you used the assumptions that the preference relation is continuous and strictly increasing.

(b) Give an example of a preference relation that is continuous but not strictly in- creasing, and for which u does not constitute a utility representation. Justify your answer.

(c) Give an example of a preference relation that is not continuous (it may or may not be strictly increasing) for which u does not constitute a utility representation. Justify your answer.

19 Allen

Question II.1 Fall 2005 majors

This question deals with some various types of possible nonconvexities in microe- conomic theory. Throughout the question, assume that we have an economy with ℓ commodities and n consumers ($i = 1, \ldots, n$), each with consumption set $\mathbb{R}^{\ell}_{+}$, initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$, and preferences $\preceq_i$, defined on $\mathbb{I}R^{\ell}_{+}$, which are strictly monotone continuous complete preorders. (Assume no free disposal.)

(a) Define competitive equilibrium in this pure exchange economy.

(b) Define the set of strongly Pareto optimal allocations.

(c) Is the set of Pareto optimal allocations convex? Briefly justify your answer.

(d) What are the properties satisfied by the aggregate excess demand correspondence restricted to $\Delta^c = \left\{p \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} p_j = 1 \text{ and } p_j \geq \epsilon \text{ for all } j = 1, \ldots, \ell\right\}$ for $\epsilon \in (0, 1/\ell)$ in this pure exchange economy?

(e) What are the properties (nonemptyness, closedness, openness, boundedness, com- pactness, convexity) satisfied by the set of competitive equilibrium prices in this pure exchange economy? State a sufficient condition for convexity of the set of competitive equilibrium prices and briefly explain your answer.

(f) Now assume that the preference relations $\preceq_i$ are weakly convex. Are they neces- sarily convex? Why or why not (briefly, in a sentence or two)?

(g) Finally, assume that each $\preceq_i$ is convex but each consumer is restricted to some subset X_i (of $\mathbb{R}^{\ell}_{+}$) which is nonempty and closed but not necessarily convex. What properties are satisfied by aggregate excess demand restricted to Δ^{ϵ} (as defined in part (d) above) for $\epsilon \in (0, 1/\ell)$?

20 Werner

Question II.2 Fall 2005 majors

Consider a reflexive, transitive, and complete preference relation on $\mathbb{R}^n_+$. For every con- sumption bundle $x \in \mathbb{R}^n_+$, let $u(x) \in \mathbb{R}_+$ be a number (if it exists) such that x is indifferent to $u(x)$ e, where e denotes the unit vector in $\mathbb{R}^n_+$.

(a) Show that, if the preference relation is continuous and strictly increasing (also called strongly monotone), then u is a well-defined function on R^n_+ and consti- tutes a utility representation of the preference relation. Clearly indicate where in your proof you used the assumptions that the preference relation is continuous and strictly increasing.

(b) Give an example of a preference relation that is continuous but not strictly increasing, and for which u does not constitute a utility representation. Justify your answer.

(c) Give an example of a preference relation that is not continuous (it may or may not be strictly increasing) for which u does not constitute a utility representation. Justify your answer.

21 Rustichini

Question III.1 Fall 2005 majors

Let $G = (S_1, \ldots, S_I, u_1, \ldots, u_I)$ be a normal form game.

(a) State (but do not prove) Kakutani's fixed point theorem.

(b) State the definition of a Nash equilibrium.

(c) Taking Kakutani's theorem as given, prove that G has a Nash equilibrium.

22 Rustichini

Question III.2 Fall 2005 majors

There are 100 people who would like to sell a used car, and 100 people who would be in- terested in buying one. Cars are either Good or Bad. The quality of a car is known by the seller. A buyer cannot determine the quality of any particular car, but does know how many cars of each type are being marketed. The reservation price of a potential seller is \$1000 if the car is Bad and \$3000 if the car is Good. The reservation price of a potential buyer is \$2000 if the car is Bad and \$4000 if the car is good. (For a car of un- known quality the buyer is willing to pay $(\ 1 - q\ )\ \$2000 + q\$4000$ where q is the probability that it is Good.) Let n be the number of cars, out of the 100, that are Good. Assume that n is common knowledge.

(a) When $n = 1$, for which prices p of used cars will the market clear?

(b) When $n = 99$, for which prices p of used cars will the market clear?

(c) For which values of n are there market equilibria in which Good cars are traded?

23 Rustichini

Question IV.1 Fall 2005 majors

Consider an infinitely repeated game with discount δ in the interval $(0, 1)$ with the fol- lowing two players state game:

	L	R
T	$(3, 1)$	$(0, 0)$
B	$(1, 1)$	$(1, 3)$

(a) Compute the set of feasible and incentive compatible payoffs of the Repeated Game. Indicate clearly the minimax values. Prove your statements.

(b) Describe a non-empty subset of the set of Subgame Perfect Equilibrium payoffs of the Repeated Game, and prove your answer.

24 Rustichini

Question IV.2 Fall 2005 majors

Consider an infinitely repeated game with discount δ in the interval $(0,1)$

(a) Define the Subgame Perfect Equilibrium Operator SPE^{δ} from closed subsets of the set of feasible and incentive compatible payoffs to subsets of the same set

(b) Prove that if A is a compact subset of the set of feasible and incentive compatible payoffs, then $SPE^{\delta}(A)$ is also compact. Do not omit any detail.

(c) Define what is as set which is self-generated by SPE^{δ}. What does this property imply?

25 Werner

Question I.1 Spring 2006 majors

Consider preference relation $\succeq$ defined on $\mathbb{R}_+^n$, where $n > 1$, by

$$x \succeq x' \text{ if and only if } x_1 \geqq x_1' \text{ and } \sum_{k=1}^{n} x_k \geqq \sum_{k=1}^{n} x_k'$$

for every $x, x' \in \mathbb{R}_+^n$. Note that this is not a lexicographic preference.

(a) Show that preference relation $\succeq$ is continuous and convex.

(b) Show that $\succeq$ does not have a utility representation.

(c) Let the demand for preference $\succeq$ at a price vector $p \in \mathbb{R}_{++}^n$ and an income $w > 0$ be defined by $\{x \in B(p,w) :$ there is no $x' \in B(p,w)$ with $x' \succ x\}$ where $B(p,w)$ denotes the budget set. Derive the demand in the case of two goods (i.e., $n = 2$). Is the demand a continuous correspondence on the domain of strictly positive prices $\mathbb{R}_{++}^2$, for arbitrary fixed income $w > 0$? Justify your answer.

26 Werner

Question I.2 Spring 2006 majors

Consider a preference relation $\succeq$ defined on the set of non-negative contingent claims R_+^S by a continuous and strictly increasing utility function $U : \mathbb{R}_+^S \to \mathbb{R}$ in the sense that $x \succeq x'$ if and only if $U(x) \geqq U(x')$ .S is a finite number of states of nature. [U strictly increasing means that if $x \geqq x'$ and $x \neq x'$, then $U(x) > U(x')$, for every $x, x' \in R_+^S$.]

(a) State necessary and sufficient condition(s) for $\succeq$ to have a state-separable util- ity representation when $S \geqq 3$

(b) Prove that the condition you stated in (a) is indeed necessary.

(c) Does the condition you stated in (a) remain necessary and sufficient in the case of two states, i.e., $S = 2$? Justify your answer.

27 Allen

Question II.1 Spring 2006 majors

This question concerns the first welfare theorem in a pure exchange economy with n con- sumers $(i = 1, \dots, n)$ and ℓ commodities. Each consumer has consumption set $X_i \subseteq \mathbb{R}^\ell_+$ and preference preorder $\preceq i$ on X_i which, for each $i = 1, \dots, n$, is assumed to be com- plete, continuous, and monotone in the sense that if $x', x'' \in X_i, x' \leqq x''$ but $x' \neq x''$ then $x' \prec_i x''$. Assume also that each consumer i has an initial endowment vector $e_i \in \mathbb{R}^\ell_+$ such that $e_i \in \operatorname{int}(X_i)$

(a) First assume that $X_i = \mathbb{R}^\ell_+$.

(i) Define competitive equilibrium for this economy.

(ii) Define Pareto optimality for this economy.

(iii) State and prove the first welfare theorem.

(b) Now assume that $X_i \subseteq \mathbb{R}^\ell_+$ is closed.

(iv) Would your answers to (i) and (ii) above (the definitions of competitive equilibrium and Pareto optimality) be changed? If so, state the new defini- tions.

(v) Does the first welfare theorem hold now, for this economy in which X_i is a closed subset of $\mathbb{R}^\ell_+$ (i.e., does this economy satisfy the hypotheses of the first welfare theorem)? Why or why not?

28 Allen

Question II.2 Spring 2006 majors

Microeconomic theory sometimes postulates in the model that an exchange economy has uncountably many consumers.

(a) Write down such a model formally.

(b) Discuss the economic interpretation of having uncountably many traders.
(c) Discuss several situations in which such models would be useful in economic the- ory and their economic motivation

(d) How are our standard results about the existence of competitive equilibrium in pure exchange economies changed (if they are changed) when we replace the as- sumption that the economy has a set $\{1, 2, \dots, n\}$ of consumers by the assumption that the economy has an uncountable set $[0, 1]$ of consumers? Briefly explain the reasons for these changes (if there are any).

29 Rustichini

Question III.1 Spring 2006 majors

Consider the general two player 2×2 game

$$\begin{array}{ccc} & L & R \\ T & (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ B & (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{array}$$

(a)
Give an example of such a game with a Nash equilibrium that is not a perfect equilibrium.

(b) Assuming that $a_{11} \neq a_{21}, a_{12} \neq a_{22}, b_{11} \neq b_{12}$, and $b_{21} \neq b_{22}$, give conditions under which the game: (i) has a single pure equilibrium; (ii) has a single mixed equilibrium; (iii) has multiple equilibria.

(c) Prove that if $a_{11} \neq a_{21}, a_{12} \neq a_{22}, b_{11} \neq b_{12}$, and $b_{21} \neq b_{22}$, then all Nash equilibria are perfect. (You may use well known results concerning Nash and perfect equilibrium so long as they are stated clearly and correctly.)

30 Allen

Question III.2 Spring 2006 majors

Consider an economy with one private good and one public good. There are two con- sumers and one firm. Consumers' utility functions are

$$u^i\left(x^i, z\right) = x^i + a_i\sqrt{z}, \quad i = 1, 2$$

for $a_1 > a_2 > 0$, where $x^i \geq 0$ denotes consumption of private good and $z \geq 0$ consump- tion of public good. The firm produces public good using private good as input accord- ing to the production function $z = \sqrt{y}$ for $y \geq 0$. Initial endowments of the private good are $\omega_x^i = 5$ and ownership shares are $\theta^i = 1/2$ for $i = 1, 2$. There are no endowments of the public good.

(a) Suppose that $a_1 = 4$ and $a_2 = 2$. Show that there is a Walrasian equilibrium (with private provision of public good) with prices $p_z = 2$ of the public good and $p_x = 1$ of the private good. Find the equilibrium allocation.

(b) Again, suppose that $a_1 = 4$ and $a_2 = 2$. Show that all Pareto optimal allo- cations in this economy have the same level z^* of public good. Derive a single equation that fully characterizes z^* (numerical soultion is not required). Is the equilibrium allocation from (a) Pareto optimal?

(c) State a definition of Lindahl equilibrium for this economy. Show that personalized prices p_z^1 and p_z^2 of the public good in an interior Lindahl equilibrium satisfy

$$\frac{p_z^1}{p_z^2} = \frac{a_1}{a_2}$$

31 Rustichini

Question IV.1 Spring 2006 majors

Consider a finite two player matrix game where the sum of the payoffs of the two players is zero.

(a) Consider first the sequential game where the first player chooses a mixed strategy (that is, a probability over actions), this strategy is communicated to the other player, who then chooses his mixed strategy. Either prove or give a counterexam- ple to the following statement: if $\hat{x}^1$ is an optimal mixed strategy for the maximizing player (who is player number 1) in the game where he moves first, it is also optimal in the game where he moves second.

(b) Let E be the set of equilibria of the of the simultaneous move game. Prove that the subset of the set of strategies of player 1 defined by: $\left\{x^1 : \exists \left(x^1, x^2\right) \in E(G) \text{ for some strategy } x^2 \text{ of player } 2\right\}$ is a convex set.

32 Rustichini

Question IV.2 Spring 2006 majors

Consider a finite extensive form game. Do not assume that game is linear, or perfect recall.

(a) Define behavioral and mixed strategies in the extensive form game.

(b) Define a notion of equivalence between strategies.

(c) Give an example to show that in an extensive form game a behavioral strategy may not have an equivalent mixed strategy

33 Werner

Question I.1 Fall 2006 majors

Consider a competitive, profit-maximizing firm with production set $Y \subset \mathbb{R}^n$ Assume that set Y is closed and bounded, and that $0 \in Y$. Let π^* be the profit function and s^* the supply correspondence of the firm.

(a) Show that function π^* is continuous and convex.

(b) Show that correspondence s^* has closed graph.

(c) Assuming that π^* is differentiable at $p \in \mathbb{R}^n$, show that

$$D\pi^*(p) = s(p)$$

(d) Assuming that π^* is differentiable at $p \in \mathbb{R}^n$ and also that s^* is differentiable at p, prove the following comparative statics property of supply:

$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$

for every $i = 1, \ldots, n$ You may use known results in mathematics without proofs, as long as you clearly state these results.

34 Werner

Question I.2 Fall 2006 majors

Consider an environment under uncertainty with S states of nature. Probabilities of states are $\pi_s > 0$ for each s, assumed known. Let $U : \mathbb{R}^S_+ \to \mathbb{R}$ be a continuous and strictly increasing utility function so that $U(c)$ is the utility of state-contingent consumption plan $c \in \mathbb{R}^S_+$

(a) State a necessary and sufficient condition for U to have a state-separable rep- resentation. (Proof is not asked for.)
The utility function U is said to be (weakly) risk averse with respect to π if $U(c) \leq U(\mathbf{E}(c))$ for every $c \in \mathbb{R}^S_+$. Here $\mathbf{E}(c)$ denotes the deterministic consumption plan equal to the expected value of c under π in every state.

(b) Prove that U has a concave expected utility representation with probabilities π_s if and only if U has a state-separable representation and is risk averse with respect to π. [You may assume in your proof that the state-separable representa- tion is differentiable.]

35 Allen

Question II.1 Fall 2006 majors

Consider a pure exchange economy with commodity space IR^ℓ (so that there are ℓ perfectly divisible commodities) and n traders $i = 1, \ldots, n$, each having consumption set $X_i \subseteq \mathbb{R}^\ell_+$ (assume that each X_i is closed), initial endowment vector $e_i \in X_i$, and preferences $\preceq_i$ (defined on X_i) which are assumed to be a complete continuous preorder.

(a) Define competitive equilibrium for this economy.

Assume that, for all $i = 1, \ldots, n$, $\preceq_i$ is locally nonsatiated.

(b) Define locally nonsatiated. The remainder (and major portion) of this question concerns the statement "Non- convexities cause problems for the existence of competitive equilibrium because they re- sult in discontinuities." It is essentially a "true or false and explain why" question, but you are asked to give a precise explanation.

(c) Identify the various ways in which the nonconvexities can arise in this economy. Briefly discuss the economic interpretation of each.

(d) For EACH possible type of nonconvexity, is the statement true or false?
(•) If it is true, clearly explain why: show (perhaps by a clear example or a clearly drawn picture) how the discontinuity arises AND then explain how the discontinuity can lead to nonexistence of equilibrium.
(•) If the statement if false, explain why: state and prove your continuity claim (or give a counterexample, clear picture, or precise explanation) and then either explain where this continuity is used in a standard proof for the existence of equilibrium or explain why existence of equilibrium doesn't require such continuity.

(e) For EACH possible type of nonconvexity, do the resulting problems (if any) with nonexistence of equilibrium disappear in a large economy? Explain.

36 Allen

Question II.2 Fall 2006 majors

Consider a smooth pure exchange economy with ℓ perfectly divisible commodities and n traders $i = 1, \ldots, n$, each with consumption set $\mathbb{R}^{\ell}_{++}$, initial endowment $e_i \in \mathbb{R}^{\ell}_{++}$, and utility function $u_i : R^{\ell}_{++} \to \mathbb{R}$ assumed to be twice continuously differentiable $\left(C^2\right)$, differentiably strictly monotone $\left(Du_i(x) \gg 0 \text{ for all } x \in \mathbb{R}^{\ell}_{++}\right)$, differentiably strictly concave $\left(D^2u(x) \text{ negative definite for all } x \in \mathbb{R}^{\ell}_{++}\right)$, and satisfy a boundary condition that for all $x \in \mathbb{R}^{\ell}_{++}$

$$cl_{\boldsymbol{R}^e} \left\{ y \in \mathbb{R}^{\ell}_{++} | u_i(y) \geq u_i(x) \right\} \cap \partial \mathbb{R}^{\ell}_{++} = \emptyset$$

(a) Describe the generic approach; be sure to discuss its motivation, the definition(s) of genericity, and state precisely its major results.

(b) Now suppose that we replace the assumption that the utilities u_i are (differentiably) strictly concave by the assumption that each $u_i : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$ is a concave func

(i) Define concavity of a continuous function $u_i : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$

(ii) Let $\preceq_i$ be the preorder (on $\mathbb{R}^{\ell}_{++}$) represented by u_i. What properties must

$\preceq_i$ satisfy? For each property, briefly explain why.

(iii) Define weak convexity, convexity, and strict convexity of a continuous complete preorder $\preceq$ on $\mathbb{R}^{\ell}_{++}$
(iv) Do the generic results that you discussed in part (a) continue to hold when utilities are concave rather (differentiably) strictly concave? For each such result, briefly explain why or why not.

(v) Consider the demand relation of a consumer with a C^2 (differentiably) strictly monotone and concave utility function. What properties does it satisfy for all prices $p \in \Delta$? What properties does it satisfy for a generic subset of prices in Δ? Justify your answer. (Hint: Draw a picture of some indifference curves corresponding to a concave utility function with some typical budget sets.)

(vi) Would your answer to (v) change if we were only interested in the demand relations of consumers for a generic subset of initial endowment vectors in $\bar{R}^{\ell}_{++}$? Explain why or why not.

37 Rustichini

Question III.1 Fall 2006 majors

Consider the set of perfect equilibria of a normal form game.

(a) Prove: a perfect equilibrium is a Nash equilibrium.

(b) Prove or disprove: the set of perfect equilibria is a closed set.

38 Rustichini

Question III.2 Fall 2006 majors

Consider the two-player game in which each of the two players states a number in the set $\{1, \ldots, M\}$. If the two players state the same number, then player 1 pays 1 dollar to player 2; otherwise player 2 pays 1 dollar to player 1 Compute the Nash equilibria of the game (where utils are equal to dollars).

Question III.3 Fall 2006 majors

Consider a normal form game $\left(I, \left(A^i\right)_{i \in I}, \left(u^i\right)_{i \in I}\right)$. Take any player k. Let b be a strictly positive real number, and c any function from $A^{-k} \to R$. Consider now the new game where all players but k have the same utility as before, but the utility of k is

$$v\left(a^k, a^{-k}\right) \equiv c\left(a^{-k}\right) + bu^k\left(a^k, a^{-k}\right)$$

for every $\left(a^k, a^{-k}\right)$. Prove or disprove the following statement: "Player k has the same best response function in the two games."

39 Rustichini

Question IV.1 Fall 2006 majors

Consider the following two-player, two-period game:

(1) In the first period, they play the game:

Table 1: First Period Game

	L	R
T	$(3, 3)$	$(1, 4)$
B	$(4, 1)$	$(2, 2)$

(2) In the second period, they play the game:

Table 2: Second Period Game

	L	R
T	$(2, 2)$	$(0, 0)$
B	$(0, 0)$	$(1, 1)$

(3) The final payoff is the sum of the payoffs in the two periods

(4) At the end of the first period, the players are informed of the action profile in that period Find the set of all the subgame perfect equilibria of the game.

40 Rustichini

Question IV.2 Fall 2006 majors

Consider a repeated game with discounting.

(a) Define the sets F of feasible payoffs, and F^* of feasible and individually rational payoffs.

(b) Prove that for every payoff $x \in F^*$, for some discount factor δ_0, for all $\delta \geq \delta_0$ there is a sequence of action profiles such that the payoff profile in the discounted game is exactly x.

(c) Can you give the values of δ_0 for which the conclusion is true?

41 Werner

Question I.1 Spring 2007 majors

Let $\succeq$ be a transitive and complete preference relation on the consumption set $X = IR_+^L$. Consider two conditions often used as alternative definitions of continuity of $\succeq$

C1. For every sequences $\{x^n\}$ and $\{y^n\}$ in X such that $\lim_n x^n = x, \lim_n y^n = y$ and $x^n \succeq y^n$, it holds $x \succeq y$

C2. For every $x \in X$, the preferred-to-x set $\{y \in X : y \succeq x\}$ and the lower contour set $\{y \in X : x \succeq y\}$ are closed. Prove that conditions C1 and C2 are equivalent.

42 Werner

Question I.2 Spring 2007 majors

Consider an agent whose preferences over state-contingent consumption plans on a set of S states ($S > 1$) have an expected utility representation $E[v(c)]$, for some probabilities of states and a von Neumann-Morgenstern (or Bernoulli) utility function $v : \mathbb{R} \to \mathbb{R}$. Assume that utility function v is strictly increasing and twice-differentiable.

Let $\rho(w, z)$ denote the risk compensation for risky claim $z \in \mathbb{R}^S$ with $E(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w

(i) Prove that A is a weakly decreasing function of w if and only if risk compensation ρ is a weakly decreasing function of w for every z with $E(z) = 0$

(ii) Prove that the negative-exponential utility function $v(x) = -e^{-\alpha x}$ for $\alpha > 0$ is, up to an increasing linear transformation, the only von Neumann-Morgenstern utility function for which risk compensation $\rho(w, z)$ is independent of w for every z with $E(z) = 0$
If you use the Theorem of Pratt, you need to state it clearly, but you are not asked to prove it.

43 Werner

Question II.1 Spring 2007 majors

Consider an agent whose preferences over state-contingent consumption plans on a set of S states ($S > 1$) have an expected utility representation $E[v(c)]$, for some probabilities of states and a von Neumann-Morgenstern (or Bernoulli) utility function $v : \mathbb{R} \to \mathbb{R}$. Assume that utility function v is strictly increasing and twice-differentiable. Let $\rho(w, z)$ denote the risk compensation for risky claim $z \in \mathbb{R}^S$ with $E(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w

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44 Allen

Question II.2 Spring 2007 majors

This question concerns smooth pure exchange economies with n traders, $i = 1, 2, \ldots, n$ and ℓ commodities. Each consumer i has preferences $\precsim i$ (assumed to be complete preorders) and initial endowment $e_i \in R^{\ell}_{+}$

(a) What additional assumptions on preferences and endowments guarantee that aggregate excess demand $Z : \Delta \to \mathbb{R}^{\ell}$ is continuously differentiable (C^1), where

$$\Delta = \left\{ p \in \mathbb{R}^{\ell}_{++} | \sum_{k=1}^{\ell} p_k = 1 \right\}?$$

(b) Define the generic approach and discuss its economic interest.

(c) If Z is C^1, what properties does the equilibrium price correspondence satisfy? Briefly discuss the economic implications of these properties.

(d) If the economies satify your assumptions in (a) so that Z is C^1, what properties are satisfied for generic profiles $e = (e_1, \ldots, e_n) \in \mathbb{R}^{\ell n}_{+}$ of initial endowments? Briefly discuss the economic implications of this.

(e) How would your answer to (a) be changed if Z were merely required to be continuous $\left(C^0\right)$?

(f) Then how would your answer to (c) change if Z were assumed only to be C^0? Prove your claim.

(g) Given your answer to (e) [in place of (a)], how would your answer to (d) be changed if Z were assumed to be C^0 but not necessarily C^1?

45 Rustichini

Question III.1 Spring 2007 majors

Consider extensive form games that are finite (that is, that have a finite set of nodes).

(a) Define an extensive form linear game.

(b) Prove that for any linear game, any player in the game, and any behavioral strat- egy of the player there is a mixed strategy of the same player that induces the same probability distribution on final nodes for any pure strategy of the other players.

(c) Give an example to show that in a linear game for a mixed strategy of the player there may be no behavioral strategy that induces the same distribution on final nodes for some pure strategy of the other players.

46 Rustichini

Question III.2 Spring 2007 majors

Consider extensive form games that are finite (that is, that have a finite set of nodes).

(a) Define an extensive form linear game.

(b) Prove that for any linear game, any player in the game, and any behavioral strat- egy of the player there is a mixed strategy of the same player that induces the same probability distribution on final nodes for any pure strategy of the other players.

(c) Give an example to show that in a linear game for a mixed strategy of the player there may be no behavioral strategy that induces the same distribution on final nodes for some pure strategy of the other players.

47 Rustichini

Question III.3 Spring 2007 majors

Define the set of correlated equilibria of a normal form game. Then consider the two players game with payoffs given by

	L	R
T	$(5,5)$	$(1,7)$
B	$(7,1)$	$(-2,-2)$

(a) Characterize (that is, describe explicitly) the set of correlated equilibria of the game

(b) Find the set of efficient correlated equilibria, that is the set of equilibria that maximize some weighted sum of the payoffs of the two players, for non-negative weights.

48 Rustichini

Question IV.1 Spring 2007 majors

A discounted repeated game has discount factor $\delta \in [0,1)$, and the discounted sum of the payoffs is normalized by the factor $1-\delta$, so that the final payoff to player i is given by

$$(1-\delta)\sum_{\tau=0}^{\infty}\delta^{\tau}u^{i}(a_{\tau})$$

where u^i is the stage game payoff to player i Consider the two stage games:

	L	R
T	$(4,4)$	$(1,6)$
B	$(6,1)$	$(2,2)$

and

	L	R
T	$(1,4)$	$(0,0)$
B	$(0,0)$	$(4,1)$

Each of these two stage games defines a repeated game. For each of these two repeated games:

(a) Prove that there exists a largest δ (possibly zero) for which the only sub-game perfect equilibrium payoff of the repeated game coincides with the set of Nash equilibrium payoff of the stage game

(b) Find the largest δ with this property for the two games. Prove your answer in detail.

49 Rustichini

Question IV.2 Spring 2007 majors

Consider the repeated game with n players, where each player i can choose a quantity $q^i \geq 0$ and profit functions are

$$u^i\left(q^1, \ldots, q^n\right) = q^i \left(\max\left\{1 - \sum_{i=1}^{n} q^i, 0\right\} - c\right)$$

(a) Find the Nash Equilibrium of the stage game.

(b) Find the value of the worst sub-game perfect equilibrium payoff for player i, for a given discount δ. Prove your answer.

50 Werner

Question I.1 Fall 2007 majors

Let $\succeq$ be a reflexive, transitive, complete and strictly increasing (i.e., strongly monotone) preference relation on the consumption set $X = IR_+^L$. Preference relation $\succeq$ is said to be homothetic if the following holds for every $x, x' \in \mathbb{R}_+^L$ and every $\lambda > 0$: if $x \sim x'$ then $\lambda x \sim \lambda x'$ where $\sim$ is the indifference relation of $\succeq$.

Prove that $\succeq$ is homothetic if and only if it there exists a utility representation u of $\succeq$ such that u is homogeneous of degree 1

51 Werner

Question I.2 Fall 2007 majors

Consider two real-valued random variables Y and Z on a probability space. You may think about Y and Z as two contingent claims on a state space. You may assume that the state space is finite. Suppose that Y can take only one of two possible values y_1, y_2 with respective probabilities $\pi_1 > 0$ and $\pi_2 > 0$ such that $\pi_1 + \pi_2 = 1$. Suppose further that the expectations of Z conditional on $\{Y = y_1\}$ and $\{Y = y_2\}$ are zero, that is $E[Z|Y = y_1] = 0$ and $E[Z|Y = y_2] = 0$

(a) Prove that $Y + Z$ is more risky (in the sense of second-order stochastic domi- nance) than Y.

(b) Prove that $Y + 2Z$ is more risky than $Y + Z$

52 Allen

Question II.1 Fall 2007 majors

This question applies to pure exchange economies with ℓ commodities and n traders, $i = 1, \ldots, n$, each having initial endowment vector $e_i \in IR^\ell$ and preferences $\preceq_i$ which are assumed throughout to be continuous complete preorders on the consumption set $X_i \subseteq \mathbb{R}^\ell$

(a) State the first welfare theorem.

(b) Prove the first welfare theorem.

(c) Does the conclusion of the first welfare theorem hold when the following complications separately (one at a time) are present?

Justify each answer by one of the following methods: pointing out that it doesn't affect your proof, explaining how your proof can be modified to encompass the complication, providing a counterexample to the conclusion of the first welfare theorem when this one complication is present, or explaining precisely how the complication prevents your proof from being modified to demonstrate that the first welfare theorem holds despite the complication. The complications are as follows:

(i) preferences that are convex but not strictly/strongly convex

(ii) preferences that are weakly convex but not convex

(iii) preferences that are nonsatiated but not locally nonsatiated

(iv) preferences that are strictly monotone but consumption sets X_i are not necessarily convex

(d) Briefly discuss (i.e., in an essay of $50 - 300$ words) the economic significance of the first welfare theorem.

53 Allen

Question II.2 Fall 2007 majors

Consider a pure exchange economy with ℓ commodities and n consumers, $i = 1, \ldots, n$, each having initial endowment vector $e_i \in \mathbb{R}^\ell$ and preferences $\preceq_i$ defined on the consumption set $\mathbb{R}^\ell_+$. Each $\preceq_i$ is assumed to be a continuous complete preorder which satisfies strict convexity and strict monotonicity. (a) What can be said about the aggregate excess demand Z of this economy? (I.e., state the theorem of Sonnenschein et al and be sure to specify the domain and range of the mapping Z.)

(b) For each property of Z you state in part (a), identify which assumption(s) on preferences are needed for the property and then prove that the assumption(s) you identify imply the property.

(c) Briefly discuss (i.e., in an essay of 50-300 words) the economic significance of the characterization of aggregate excess demand.

54 Rustichini

Question III.1 Fall 2007 majors

(a) Show that every finite game possesses a Nash equilibrium in which no player places a strictly positive probability on a weakly dominated strategy.

(b) Improve this result from (a) by showing that every finite game possesses a Nash equilibrium σ in which for every player i, σ_i is not dominated.

(c) Show by an example that the result in the previous point (b) requires finiteness.

55 Rustichini

Question III.2 Fall 2007 majors

(a) Define the set of correlated strategies and correlated equilibria for a finite game.

(b) Define an augmented game, and show how the Bayesian-Nash equilibrium of the augmented game and the correlated equilibrium are related.

(c) Prove that the set of correlated equilibrium payoffs is a closed, convex, non-empty set.

56 Rustichini

Question IV.1 Fall 2007 majors

(a) Define the Nash equilibrium operator NE^6 and the Sub-game perfect Equilibrium operator SP^δ from subsets of the set of feasible and incentive compatible payoffs to subsets of the same set of payoffs.

(b) Define a self-generated set for each of the two operators in part (a).

(c) Show an example where the two operators are different.

57 Rustichini

Question IV.2 Fall 2007 majors

(a) Define the set of correlated strategies and correlated equilibria for a finite game.

(b) Define an augmented game, and show how the Bayesian-Nash equilibrium of the augmented game and the correlated equilibrium are related.

(c) Prove that the set of correlated equilibrium payoffs is a closed, convex, non-empty set.

58 Werner

Question I.1 Spring 2008 majors

Consider two real-valued random variables y and z with the same expectations, $E(y) = E(z)$. Answer the following question (a):

(a) Show that if y is more risky than z, then $\text{var}(y) \geq \text{var}(z)$, where $\text{var}(y)$ denotes the variance of y Answer either one of questions (b) or (c), but not both.

(b) Show that, if z is more risky than y and y is more risky than z, then y and z have the same distribution, i.e., $F_y(t) = F_z(t)$ for every t, where F_y and F_z denote the cumulative distribution functions of y and z. You may assume that y and z take only finitely many values.

(c) Give an example of random variables y and z (with the same expectations) such that neither y is more risky than z nor z is more risky than y.

59 Werner

Question I.2 Spring 2008 majors

Consider a finite set $\{p^t, x^t\}_{t=1}^T$ of pairs of price vectors $p^t \in \mathbb{R}^l_{++}$ and consumption Ebundles $x^t \in \mathbb{R}^l_+$. Utility function $u : \mathbb{R}^l_+ \to \mathbb{R}$ is said to rationalize this set if $u\left(x^t\right) \geq u(x)$ Evor every $x \in \mathbb{R}^l_+$ such that $p^t x^t \geq p^t x$

(a) State the Generalized Weak Axiom of Revealed Preference (GWARP) for $\{p^t, x^t\}_{t=1}^T$.

(b) Show that if a locally non-satiated utility function u rationalizes $\{p^t, x^t\}_{t-1}^T$, then the GWARP holds.

(c) Show that the assumption of local non-satiation in (b) cannot be dispensed with. That is, provide an example of a utility function that is locally satiated and rationalizes a set of pairs of prices and consumption bundles that violates the GWARP.

(d) State the Theorem of Afriat providing necessary and sufficient conditions for a set of pairs of prices and consumption bundles to be rationalized by a locally non-satiated utility function. A proof is not asked for.

60 Allen

Question II.1 Spring 2008 majors

Consider a pure exchange economy with n consumers $i = 1, \ldots, n$, each having initial endowment $e_i \in \mathbb{R}^\ell_+$ and preferences $\preceq_i$ assumed to be complete preorders on $\mathbb{R}^\ell_+$.

(a) Define the core of this economy.

(b) State and prove the Equal Treatment Property for the core of a pure exchange economy.

(c) State the Debreu-Scarf Theorem and discuss its economic significance.

61 Allen

Question II.2 Spring 2008 majors

Consider a pure exchange economy with N consumers $(i = 1, \ldots, N)$ and L commodities. Consumer i has an initial endowment vector $e_i \in R_+^L$ and a rational preference $\succsim_i$ (complete, reflexive, and transitive binary relation, i.e. complete preorder) on the consumption set $X_i \subset R_+^L$. You may assume that the preferences are continuous.

(a) Define competitive (Walrasian) equilibrium and Pareto efficient (optimal) allocation in this pure exchange economy.

(b) State and prove the first welfare theorem. If you need any extra assumption(s), make sure to state all of them formally and clearly.

(c) Present one example of pure exchange economy where the first welfare theorem fails to hold. Explain which assumption you used in (b) is violated.

62 Allen

Question II.3 Spring 2008 majors

Consider a pure exchange economy with two consumers $i = 1, 2$ and L commodities. Consumer i' s preference can be represented by a function $u_i(x_i) = \sum_{l=1}^{L} \ln(x_{i,l} + 1)$ Answer the following questions.

(a) Suppose that the initial endowment vectors are given by $e_1 = e_2 = (1, \ldots, 1) \in \mathbb{R}_+^L$ Define Pareto efficient allocations in this economy and prove that this initial allocation is Pareto-efficient (Drawing a graph is not enough).

(b) Define competitive equilibrium in this economy and find a competitive equilibrium given these endowment vectors. Also prove that there is no other competitive equilibrium.

(c) Prove that there exists a competitive equilibrium for any pair of initial endowment vectors $(e_1, e_2) \in \mathbb{R}_+^{2L}$ (If you cannot prove it for general L, then you may prove it for $L = 2$ to receive a partial credit).

63 Rustichini

Question III.1 Spring 2008 majors

Two players have to state an integer number between 100 and 200 simultaneously. The payoff to each of the two players is equal to the minimum of the statements. In addition, if one of the two made a statement strictly larger than the other, then he pays an amount R to the other. No transfer is made if the two statements match.

(a) Find the set of all Nash Equilibria (in mixed strategies) for any $R > 0$

(b) Find the set of rationalizable strategies for any $R > 0$

(c) Prove your answer precisely.

64 Rustichini

Question III.2 Spring 2008 majors

For a finite normal form game:

(a) Define the following property: an action is ”never a best response”.

(b) Prove that an action is never a best response if and only if it not strictly dominated by a mixed strategy.

65 Rustichini

Question IV.1 Spring 2008 majors

Let G be a finite normal form game (that is, finite number of players and actions), and consider the repeated game with a finite, fixed and known number of rounds, T. Call this game $G(T)$

(a) Prove that if the stage game G is two-person zero-sum then in any equilibrium of $G(T)$ players play in every period a strategy profile which is a Nash equilibrium of the stage game.

(b) Prove that if the stage game (not necessarily zero-sum) has a unique Nash equilibrium, then in any sub-game perfect equilibrium of $G(T)$ players play in every period a strategy profile which is a Nash equilibrium of the stage game.

(c) Is the conclusion that in any sub-game perfect equilibrium of $G(T)$ players play in every period a strategy profile which is a Nash equilibrium of the stage game, true if G is any finite normal form game? Prove your answer or show a counter-example.

66 Rustichini

Question IV.2 Spring 2008 majors

Let G be a finite normal form game (that is, finite number of players and actions), and consider the repeated game with a finite, fixed and known number of rounds, T. Call this game $G(T)$ Prove the following: If $(a_1, \ldots, a_{T^1})$ is a subgame perfect equilibrium outcome in $G\left(T^1\right)$ and $(b_1, \ldots, b_{T^2})$ is a subgame perfect equilibrium outcome in $G\left(T^2\right)$, then the patching: $(a_1, \ldots, a_{T^1}, b_1, \ldots, b_{T^2})$ is a subgame perfect equilibrium outcome in $G\left(T^1 + T^2\right)$

67 Werner

Question IV.3 Spring 2008 majors

Consider a decision maker who has a rank-dependent preference over lotteries as described in, for example, Yaari's (1987) dual theory of choice under risk. He has 1/3 unit of initial wealth, and can invest part of it in a risky asset. If he invests $x \in [0, 1/3]$ in the risky asset, and keeps the remaining $1/3 - x$ in the form of cash, his eventual wealth will be $(1/3 - x) + \theta x$, where θ is uniformly distributed in $[0, 3]$. The decision maker's problem is to maximize his utility by choosing x. Prove that his problem has corner solutions; i.e., either $x = 0$ is optimal, or $x = 1/3$ is optimal.

68 Rustichini

Question IV.4 Spring 2008 majors

Let C_2 be the set of possible period- 2 consumptions, $\mathcal{L}_2 = \Delta(C_2)$ be the set of simple lotteries over C_2, C_1 be the set of possible period-1 consumptions, and $\mathcal{L}_1 = \Delta(C_1 \times \mathcal{L}_2)$ be the set of simple lotteries over $C_1 \times \mathcal{L}_2$. Let $\succsim$ be a preference defined over $\mathcal{L}_1$, and $\{\succsim_{c_1}\}$ be a class of preferences indexed by $c_1 \in C_1$ and defined over $\mathcal{L}_2$. Assume that the preference relation $\succsim$ (defined over $\mathcal{L}_1$) is complete, transitive, and satisfies the independence axiom: THE INDEPENDENCE AXIOM: For any $L_1, L_1', L_1'' \in \mathcal{L}_1$ and $a \in (0,1)$

$$L_1 \succsim L_1' \Leftrightarrow aL_1 + (1-a)L_1'' \succsim aL_1' + (1-a)L_1''$$

Assume that, for any $c_1 \in C_1$, the preference relation $\succsim_{c_1}$ (defined over $\mathcal{L}_2$) is complete and transitive (but may not satisfy the independence axiom). Also assume that the preference relations $\succsim$ and $\succsim_{c_1}$'s satisfy the time consistency axiom:
THE TIME CONSISTENCY AXIOM: For any $c_1 \in C_1$ and $L_2, L_2' \in \mathcal{L}_2$

$$\delta_{(c_1, L_2)} \succsim \delta_{(c_1, L_2')} \Leftrightarrow L_2 \succsim_{c_1} L_2'$$

where $\delta_{(c_1, L_2)}$ denotes the degenerate lottery that puts a point mass on (c_1, L_2). Now, consider the following axioms:

AXIOM T: For any $a \in (0,1), c_1 \in C_1$, and $L_2, L_2' \in \mathcal{L}_2$

$$L_2 \sim_{c1} L_2' \Rightarrow a\delta_{(c_1, L_2)} + (1-a)\delta_{(c_1, L_2)} \sim \delta_{(c_1, aL_2 + (1-a)L_2)}$$

AXIOM B: For any $a \in (0,1), c_1 \in C_1$, and $L_2, L_2' \in \mathcal{L}_2$

$$L_2 \sim_{c_1} L_2' \Rightarrow aL_2 + (1-a)L_2' \sim_{c_1} L_2$$

AXIOM T says that, if L_2 and L_2' are equally good, then the decision maker cannot care less whether uncertainty is resolved earlier or later. AXIOM B says that, if L_2 and L_2' are equally good, then any mixture between them is also indifferent to each of those two lotteries. Prove that, given the above assumptions, AXIOM T implies AXIOM B.

69 Werner

Question I.1 Fall 2008 majors

(a) State the Theorem of Pratt asserting equivalence of three ways of comparing risk aversion of agents whose preferences over risky claims have expected utility representation: Arrow-Pratt measure of risk aversion, risk compensation, and concave transformation of the von Neumann-Morgenstern utility function. Make sure that you clearly list all assumptions of the theorem.

(b) Prove the following two parts of the theorem you stated: (i) ranking according to Arrow-Pratt measure implies ranking according to concave transformation of utility function, (ii) ranking according to concave transformation of utility function implies ranking according to risk compensation.

(c) Give an example of two von Neumann-Morgenstern utility functions v_1 and v_2 such that neither v_1 is more risk averse than v_2 nor v_2 is more risk averse than v_1 in the sense of the Theorem of Pratt.

70 Werner

Question I.2 Fall 2008 majors

Consider a finite set of observations $\{p^t, y^t\}_{t=1}^{T}$ of pairs of price vectors $p^t \in R^l_{++}$ and production plans $y^t \in \mathbb{R}^l$. We say that production set $Y \subset \mathbb{R}^l$ profit-rationalizes this set of observations if $y^t \in Y$ and $p^t y^t = \max_{y \in Y} p^t y$ for every t

(a) State the Weak Axiom of Profit Maximization (WAPM).

(b) Prove that a set of observations $(p^1, y^1), \ldots, (p^T, y^T)$ satisfies WAPM if and only if there exists a closed, convex production set Y that profit-rationalizes these observations.

71 Allen

Question II.1 Fall 2008 majors

For a pure exchange economy with ℓ commodities and n traders, (a) State the second welfare theorem (also known as the second fundamental theorem of welfare economics)

(b) Prove the theorem, and

(c) Briefly compare its statement and proof to the analogous part of the Debreu-Scarf Theorem.

72 Allen

Question II.2 Fall 2008 majors

Consider an economy with N consumers and L goods $\left(\mathcal{E}\left(\mathbb{R}^L_+, \succeq_i, e_i, i = 1, \ldots, N\right)\right)$ The preference of consumer $i = 1, \ldots, N$ can be represented by a differentiable function $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ that satisfies $Du_i(x_i) \gg 0$. Answer the following questions. (a) If u_i satisfies an additional assumption, then $x^*(\gg 0) \in A(A$ is the set of feasible allocations) is Pareto efficient if and only if x^* solves the following maximization problem for some $a \in \mathbb{R}^N_{++}$. What is this assumption?

(b) Define competitive equilibrium with transfer in this economy. Also derive a system of equations such that $(x^*, p^*) \gg 0$ is a competitive equilibrium with transfer if and only if $(x^*, p^*) \gg 0$ solves the system of equations for some multipliers (again state any additional assumption you used).

(c) Using the results from (a) and (b), prove the following version of the second welfare theorem in this economy (with an appropriate assumption on utility functions): "If $x^* \in \mathbb{R}^{L \times N}_{++}$ is Pareto efficient, then there exists a price vector $p^* \in \mathbb{R}^L_{++}$ such that (x^*, p^*) is a competitive equilibrium with transfer".

73 Allen

Question II.3 Fall 2008 majors

Consider a pure exchange economy with N consumers and L goods. Let $E : \mathbb{R}_{++}^{L\times N} \Rightarrow \mathbb{R}_{+}^{L\times N} \times \mathbb{R}_{+}^{L-1}$ be the equilibrium correspondence (normalize p_L to 1), i.e.

$E(e)$ is the set of competitive equilibria when the initial endowment vector is e (the preference is fixed). Consumer i 's preference can be represented by a contimuously differentiable (i.e. $Du_i\left(x_i\right)$ is continuous), concave function $u_i : \mathbb{R}_{+}^{L} \rightarrow \mathbb{R}$ that satisfies the following two conditions:

(1). $Du_i\left(x_i\right) \gg 0$, and

(2). If $u_i(x) \geq u_i\left(x_i'\right)$ for some $x_i' \gg 0$, then $x_i \gg 0$ for $i = 1, \ldots, N$.
Answer the following questions.

(a) Write down a system of equations to characterize the set of competitive equilibria $E(e)$ for $e \in \mathbb{R}_{++}^{L}$

(b) "E(.) is upper hemicontinuous at $e \in \mathbb{R}_{++}^{L}$ " Translate this into a precise mathematical statement regarding the equations from (a).

(c) Prove that $E(\cdot)$ is upper hemicontinuous at any $e \in \mathbb{R}_{++}^{L}$

74 Rustichini

Question III.1 Fall 2008 majors

(a) Define the correlated strategies and correlated equilibria;

(b) Prove that the set of correlated equilibria and the set correlated equilibrium payoffs are closed and convex.

(c) Give an example of a game where correlated equilibria and Nash Equilibria are the same.

(d) Give an example of a game where the set of correlated equilibria is not the convex hull of the set of Nash equilibria.

75 Rustichini

Question III.2 Fall 2008 majors

Let G be a normal form game with n players, $\left(I, \left(A^i\right)_{i\in I}, \left(u^i\right)_{i\in I} \right)$. For every i, S^i is the set of mixed strategies for i, and S the set of mixed strategy profiles. Let $m \equiv \sum_{i\in I} \# A^i$, and write a vector in R^m as $\left(x_k^i\right)_{k=1,\ldots,\# A, i=1,\ldots,n}$. For vectors $x, y \in R^m$ we write $x > y$ if for every $j = 1, \ldots, m, x^j > y^j$. Let $\eta \in R^m, \eta > 0$. define:

$$I(\eta) \equiv \{\epsilon \in R^m : \eta > \epsilon > 0\}$$

The perturbed game G_e is the game where player $i \in I$ can only use mixed strategies s^i such that for every $k, s_k^i \geq \epsilon_k^i$. The set of Nash equilibria of this game is denoted by $E\left(G_\eta\right)$ A mixed strategy profile $s \in S$ is called strictly proper equilibrium of G if there is an $\eta > 0$ and a continuous function $P : I(\eta) \to S$ such that (i) $\lim_{\epsilon,0} P(\epsilon) = s$, and (ii) for every $\epsilon, P(\epsilon) \in E\left(G_\epsilon\right)$

(a) Define a proper equilibrium

(b) Prove that a strictly proper equilibrium is a proper equilibrium.

76 Rustichini

Question IV.1 Fall 2008 majors

Let G be a normal form game with n players, $\left(I, \left(A^i\right)_{i\in I}, \left(u^i\right)_{i\in I}\right)$. For every i, S^i is the set of mixed strategies for i, and S the set of mixed strategy profiles.
Let $m \equiv \sum_{i\in I} \# A^i$, and write a vector in R^m as $\left(x_k^i\right)_{k=1,\ldots,\# A^i, i=1,\ldots,n}$. For vectors $x, y \in R^m$ we write $x > y$ if for every $j = 1, \ldots, m, x^j > y^j$. Let $\eta \in R^m, \eta > 0$. define:

$$I(\eta) \equiv \{\epsilon \in R^m : \eta > \epsilon > 0\}$$

The perturbed game G_ϵ is the game where player $i \in I$ can only use mixed strategies s^i such that for every $k, s_k^t \geq \epsilon_k^t$. The set of Nash equilibria of this game is denoted by $E\left(G_\eta\right)$ A mixed strategy profile $s \in S$ is called strictly proper equilibrium of G if there is an $\eta > 0$ and a continuous function $P : I(\eta) \to S$ such that (i) $\lim_{\epsilon,10} P(\epsilon) = s$, and (ii) for every $\epsilon, P(\epsilon) \in E\left(G_\epsilon\right)$

(a) Define a proper equilibrium

(b) Prove that a strictly proper equilibrium is a proper equilibrium.

77 Rustichini

Question IV.2 Fall 2008 majors

(a) Prove that every finite extensive form game of perfect information has at least one pure strategy subgame perfect equilibrium. (b) Give an example of a finite extensive form game having no pure strategy subgame perfect equilibrium.

78 Allen

Question IV.3 Fall 2008 majors

Consider a decision maker whose preference admits a maxmin expected-utility representation as described in, for example, Gilboa and Schmeidler (1989). There are 2 urns in front of this decision maker. In the first urn, there are 5 black balls and 5 red balls. In the second urn, there are 10 balls of the color either black or red, but the composition is unknown. Consider the following 4 bets: bet #1: win $10 if a random ball drawn from the first urn is black, and $0 if red bet #2: win $10 if a random ball drawn from the first urn is red, and $0 if black bet #3: win $10 if a random ball drawn from the second urn is black, and $0 if red bet #4: win $10 if a random ball drawn from the second urn is red, and $0 if black

(a) Is it possible that the decision maker strictly prefers bet #1 to bet #3? Explain your answer.

(b) Is it possible that the decision maker strictly prefers bet #3 to bet #1? Explain your answer.

Suppose instead the decision maker's preference admits a second-order expected-utility representation as described in, for example, Neilson (1993). Suppose the decision maker is indifferent between bets #3 and #4

(c) Is it possible that the decision maker strictly prefers bet #1 to bet #3? Explain your answer.

(d) Is it possible that the decision maker strictly prefers bet #3 to bet #1? Explain your answer.

79 Allen

Question IV.4 Fall 2008 majors

Consider a decision maker whose preference admits a maxmin expected-utility representation as described in, for example, Gilboa and Schmeidler (1989). His preference can hence be summarized by a compact and convex set of subjective probabilities and a Bernoulli utility function. Suppose that, when this decision maker receives new information, he updates his beliefs by applying Bayes' rule to each of his priors. In other words, after he receives new information, his Bernoulli utility function remains the same, while his new set of subjective probabilities is the set of conditional probabilities obtained by conditioning his original subjective probabilities on the new information.

Use an example to illustrate that such a decision maker may suffer from the dynamic consistency problem. What you need to decribe in your example are (i) a Bernoulli utility function, (ii) a state space, (iii) a signal (with multiple possible realizations) that conveys information to the decision maker regarding the true state, and (iv) two acts/bets, a and b, such that the decision maker strictly prefers a to b before observing the signal, but strictly prefers b to a afterward regardless of the realization of the signal.

80 Allen

Question I.1 Spring 2009 majors

Consider a pure exchange economy with two traders (indexed by subscripts 1 and 2) and ℓ commodities. Suppose that both initial endowment vectors, e_1 and e_2, are strictly positive $\left(e_i \in \mathbb{R}^{\ell}_{++}, i = 1, 2\right)$. Let $F = \left\{(x_1, x_2) \in \mathbb{R}^{2\ell}_{+} | x_1 + x_2 = e_1 + e_2\right\}$ denote the set of feasible allocations (with no free disposal). Suppose that each trader $i = 1, 2$ has a preference relation $\ll i$ defined on F which is representable by a utility function $u_i : F \to \mathbb{R}$.

(a) Define what it means for u_i to represent $\preccurlyeq_i$

(b) If there is a utility $u_i : F \to \mathbb{R}$ that represents $\lll_i$, what assumption must $\lll_i$ satisfy?

(c) What additional assumptions (be very precise) are needed for the first welfare theorem to hold in this economy?

(d) What minimal additional assumptions on $\preccurlyeq_1$ and $\preccurlyeq_2$ guarantee that u_1 and u_2 can always be chosen to represent $\ll 1$ and $\ll 2$ respectively such that at any Pareto optimal allocation $\hat{x} \in \mathbb{R}^{2\ell}_{+}, u_1(\hat{x}) < u_2(\hat{x})$? Can you find an example of $\preccurlyeq_1$ and $\lll_2$ on F such that there is a utility $\bar{u}_1 : F \to R$ representing $\preccurlyeq_1$ for which there is no $\bar{u}_2 : F \to R$ representing $\preccurlyeq_2$ such that $\bar{u}_1(\hat{x}) < \bar{u}_2(\hat{x})$ for every Pareto optimal allocation $\hat{x}$. Hint: Consider linear preferences of F such that $x' \sim_1 x''$ if and only if $x'_{11} + x'_{12} = x''_{11} + x''_{12}$ and $x' \sim_2 x''$ if and only if $x'_{21} + x'_{22} = x''_{21} + x''_{22}$ where $x \in F$ is written as $x = (x_{11}, x_{12}, x_{21}, x_{22})$ but suppose that person 1 is indifferent between consuming all of the economy's resources and consuming nothing.

81 Allen

Question I.2 Spring 2009 majors

In general equilibrium theory, changing from a model with a finite number of traders to one with uncountably many - in fact, an atomless continuum of traders can lead to better results on the existence of competitive equilibrium and its welfare properties. Consider a pure exchange economy in which there are ℓ commodities and each trader i has consumption set $X_i \subseteq \mathbb{R}^{\ell}_{+}$ and $e_i \in \operatorname{int}(X_i)$, so that $e_i \gg 0$. For each complication below, explain whether changing from a finite number of traders to an atomless continuum leads to either better results or simpler/easier proofs for the existence of competitive equilibrium and the first and second fundamental theorems of welfare economics. Explain your reasoning briefly. You may assume that preferences $\lll_i$ are strictly monotone complete continuous preorders defined on X_i

(a) Preferences that are convex but not strictly convex.

(b) Nonconvex preferences.

(c) Nonconvex consumption sets that are closed (and bounded from below since

$$X_i \subseteq \mathbb{R}^{\ell}_{+})$$

(d) Consumption externalities.

82 Werner

Question II.1 Spring 2009 majors

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm is taxed at rate $t > 0$ of its total cost. The firm's profit maximization problem is

$$\max_{x \geq 0} [qf(x) - wx - t(wx)]$$

Let $x^*(t)$ denote the profit maximizing vector of inputs (assumed unique) as function of tax rate t

(a) State a definition of production function f being supermodular. State a criterion for supermodularity of f under an additional assumption that f is twice differentiable.

(b) Show that if f is supermodular, then input demand x^* is a nonincreasing function of t, that is, if $t' \geq t$, then $x^*(t') \leq x^*(t)$. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

83 Werner

Question II.2 Spring 2009 majors

Consider an agent whose preferences over state-contingent consumption plans on a finite state space S have an expected utility representation with strictly increasing and twice-differentiable utility function $v : \mathbb{R} \to \mathbb{R}$ and probability measure π on 2^S. Prove that the agent is risk averse if and only if $E[v(\tilde{z})] \geq E[v(\tilde{y})]$ for every $\tilde{y} : S \to \mathbb{R}_+$ and $\tilde{z} : S \to \mathbb{R}_+$ such that $E(\tilde{y}) = E(\tilde{z})$ and $\tilde{y}$ is more risky than $\tilde{z}$. Expected value E is taken with respect to probability measure π. Your definition of more risky should be stated in terms of cumulative distribution functions of $\tilde{y}$ and $\tilde{z}$. You may use the Theorem of Pratt without proving it.

84 Rustichini

Question III.1 Spring 2009 majors

An extensive form game (EFG) is said to be linear if every information set is crossed at most once by every history.

(a) Give an example of an EFG which is not linear.

(b) Compare linear games and games with perfect recall. Is one of the two a subset of the other? Prove your answer.

85 Rustichini

Question III.2 Spring 2009 majors

Part 1: Find the Nash equilibria of the following game. (a) Two players move sequentially, an initial amount of I dollars is paid by a third part into a common fund.

(b) Player 1 and player 2 pick a card out of a set of three cards numbered 1, 2, and 3.

(c) Player 1 has decide whether he bets or folds. If player 1 folds, player 2 gets an amount of I dollars and the game is over. If player 1 bets, he has to pay B dollars to the common fund, and the game goes to the next stage.

(d) Player 2 is informed of the decision of player 1, and has to decide whether he bets or folds. If player 2 folds, player 1 gets the I dollars, and gets the B amount back. If player 2 bets, he has to pay B dollars to the common fund, and the game goes to the next stage.

(e) Both players show their card; the player with the highest number wins the amount in the common fund, that is the I amount and the amount $2 \times B$ paid in the earlier stages.
The utility of monetary amounts is linear. The equilibria may depend on the values of the parameters I and B : please specify the set as a function of these parameters.

Part 2:

Find the Nash equilibria of the game described in the first part, where the two players pick each one out of four cards, numbered $1, 2, 3$, and 4. In the final stage card 4 beats card 3 , which beats card 2 , which beats card 1.

86 Rustichini

Question IV.1 Spring 2009 majors

Suppose there are two urns, each containing a number of balls that are either Red or Blue. Suppose we are to draw one ball from each urn. There are four possibilities: RR (both balls are red), RB (the ball from the first urn is red, and the one from the second is blue), BR (the ball from the first urn is blue, and the one from the second is red), and BB (both balls are blue). Let $\{RR, RB, BR, BB\}$ be the state space.

Consider four acts, denoted $1R, 1B, 2R, 2B$, where $1R$ means betting on a red ball out of the first urn, $2R$ on a red ball out of the second urn, etc. These four acts are summarized by the following matrix:

$$\begin{array}{ccccc} & RR & RB & BR & BB \\ 1R & 1 & 1 & 0 & 0 \\ 1B & 0 & 0 & 1 & 1 \\ 2R & 1 & 0 & 1 & 0 \\ 2B & 0 & 1 & 0 & 1 \end{array}$$

Suppose an agent is indifferent between acts $1R$ and $1B$, and between acts $2R$ and 2 B. But suppose he strictly prefers act $1R$ to act $2R$.

(a) State Savage's axiom P2 (also known as the Sure Thing Principle).

(b) Explain why the agent's preference violates Savage's axiom P2.

87 Rahman

Question IV.2 Spring 2009 majors

Construct an example of a social choice function that is:

(a) strategy-proof,

(b) not Maskin-monotonic,

(c) fully/strongly implementable in dominant strategy equilibrium by some indirect mechanism,

(d) not fully/strongly implementable in dominant strategy equilibrium by the direct revelation mechanism.

Please exaplain how your example satisfied each of the above four properties.

88 Rahman

Question IV.3 Spring 2009 majors

Consider a quasilinear environment where two agents are to contribute to a public project. Let $K = \{0, 1\}$ be the possible levels of the project, with 1 meaning that the project is "done", and 0 "not done". Agent i's private valuation of the project is denoted by θ_i, which is independently drawn from a uniform distribution on $[0, 1]$. The project costs c to finish, where c is a constant strictly in between 0 and 2. Let $k : \Theta \to K$, where $\Theta = \Theta_1 \times \Theta_2 = [0, 1] \times [0, 1]$, denote the following allocation function:

$$k(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 + \theta_2 \geq c \\ 0 & \text{otherwise} \end{cases}$$

A transfer rule $t : \Theta \to \mathbb{R}^2$ specifies, for each agent i, the amount of monetary transfer received by agent i at each $\theta = (\theta_1, \theta_2) \in \Theta$. Writing $t(\theta)$ as $(t_1(\theta), t_2(\theta))$, we say that the transfer rule t balances the budget if, for any θ

$$t_1(\theta) + t_2(\theta) = \begin{cases} -c & \text{if } k(\theta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Construct a budget-balancing transfer rule t that Bayesian-implements the allocation rule k; i.e., construct a budget-balancing transfer rule t such that the social choice function (k, t) is Bayesian incentive compatible.

89 Werner

Question I.1 Fall 2009 majors

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm's profit maximization problem is

$$\max_{x \geq 0} [qf(x) - wx]$$

Let $x^*(q)$ denote the profit maximizing vector of inputs (assumed unique) as function of output price q

(a) State a definition of production function f being supermodular. Show that the Cobb-Douglas production function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, where $\alpha_i > 0$ for all i, and

$$\sum_{i=1}^{n} \alpha_i < 1, \text{ is supermodular.}$$

(b) Show that if f is supermodular, then input demand x^* is a nondecreasing function of q. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

90 Rahman

Question I.2 Fall 2009 majors

Construct an example of a social choice function that is: (a) strategy-proof,

(b) not Maskin-monotonic,

(c) fully/strongly implementable in dominant strategy equilibrium by some indirect mechanism,

(d) not fully/strongly implementable in dominant strategy equilibrium by the direct revelation mechanism.

Please exaplain how your example satisfied each of the above four properties.

91 Werner

Question II.1 Fall 2009 majors

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twice-differentiable von Neumann-Morgenstern (or Bernoulli) utility $v : R \to \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable $\tilde{z}$ with $E(\tilde{z}) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w

(a) Prove that A is an increasing function of w if and only if risk compensation ρ is an increasing function of w for every $\tilde{z}$ with $E(\tilde{z}) = 0$ and $\tilde{z} \neq 0$

(b) Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$, where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived. If you use the Theorem of Pratt, you need to state it clearly, but you are not asked to prove it.

92 Allen

Question II.2 Fall 2009 majors

This question concerns the characterization of aggregate excess demand in pure exchange economies with ℓ commodities in which n consumers, $i=1,2,\ldots,n$ have continuous ordinal utilty functions u_i defined on their consumption sets $\mathbb{R}_{+}^{\ell}$ and initial endowment vectors $e_i \in \mathbb{R}_{+}^{\ell}$. Let $p \in \mathbb{R}_{++}^{\ell}$ denote a price vector and write for i 's individual excess demand

$$z\left(\cdot; u_i, e_i\right): \mathbb{R}_{++}^{\ell} \rightarrow \mathbb{R}^{\ell}$$

and

$$Z\left(\cdot; u_1, e_1, \ldots, u_n, e_n\right): \mathbb{R}_{++}^{\ell} \rightarrow \mathbb{R}$$

for aggregate excess demand, where

$$Z\left(p; u_1, e_1, \ldots, u_n, e_n\right)=\sum_{i=1}^{n} z\left(p; u_i, e_i\right)$$

(a) Briefly explain why

$$Z\left(p; u_1, e_1, \ldots, u_n, e_n\right)=Z\left(\lambda p; u_1, e_1, \ldots, u_n, e_n\right)$$

for any strictly positive scalar $\lambda>0$
(b) Briefly explain why

$$Z\left(p; \lambda u_1, e_1, \ldots, \lambda u_n, e_n\right)=Z\left(p; u_1, e_1, \ldots, u_n, e_n\right)$$

for any strictly positive scalar $\lambda>0$
(c) Is it true that

$$Z\left(p; u_1, \lambda e_1, \ldots, u_n, \lambda e_n\right)=\lambda Z\left(p; u_1, e_1, \ldots, u_n, e_n\right)$$

for all strictly positive scalars $\lambda>0$? Briefly explain why or why not.

(d) Find an example of e_1 and e_2 such that

$$z\left(p; u_1, e_1\right)+z\left(p; u_2, e_2\right)=z\left(p; u_1+u_2, e_1+e_2\right)$$

for all u_1 and u_2 and for all $p \in \mathbb{R}_{++}^{\ell}$
(e) Find an example of u_1 and u_2 such that

$$z\left(p; u_1, e_1\right)+z\left(p; u_2, e_2\right)=z\left(p; u_1+u_2, e_1+e_2\right)$$

for all $p \in \mathbb{R}_{++}^{\ell}$ and all $e_1, e_2 \in \mathbb{R}_{+}^{\ell}$
(f) For what price vectors (specify a necessary and sufficient condition) $p_1, p_2 \in \mathbb{R}_{++}^{\ell}$ is it

true that $\left(\frac{(p_1+p_2)}{2}; u_i, e_i\right)=z\left(p_1; u_i, e_i\right)+z\left(p_2; u_i, e_i\right)$
$2z\left(\frac{1}{2}\right.$

for all u_i and all $e_i \in \mathbb{R}_{+}^{\ell}$? (g) State the theorem of Sonnenschein, Debreu, Mantel, Mas-Colell, McFadden and Richter characterizing the aggregate excess demand functions that can be generated by pure exchange economies satisfying certain assumptions (state these assumptions precisely).

(h) Suppose that we restrict all endowments $e_1=\cdots=e_n=(1,\ldots,1) \in \mathbb{R}^{\ell}$ and all utilities to be Cobb-Douglas with possibly different parameters $\left(u_i=x_1^{\text{oulf}} x_2^{\text{out}} \cdots x_{\ell}^{\alpha_{ct}},\right.$ where $\alpha_{ji}>0$ and $\sum_{j=1}^{\ell} \alpha_{ji}=1$ for all $i=1\ldots,n$). Characterize the resulting aggregate excess functions that can arise in such economies.

93 Rustichini

Question III.1 Fall 2009 majors

(a) Give an example of a game where the set of correlated equilibrium payoffs is the set of Nash equilibrium payoffs.

(b) A game with public communication is an extensive form game in which first players observe a public signal, and then they play a normal form game. Prove that the set of equilibrium payoffs is a subset of the set of correlated equilibrium payoffs. Give an example to show that the inclusion may be strict.

94 Rustichini

Question III.2 Fall 2009 majors

(a) Find the Nash equilibria of the following game.

(i) Two players move sequentially, an initial amount of I dollars is paid by a third part into a common fund.

(ii) Player 1 and player 2 pick a card out of a set of three cards numbered $1, 2,$ and-3

(iii) Player 1 has to decide whether he bets or folds. If player 1 folds, player 2 gets an amount of I dollars and the game is over. If player 1 bets, he has to pay B dollars to the common fund, and the game goes to the next stage.

(iv) Player 2 is informed of the decision of player 1, and has to decide whether he bets or folds. If player 2 folds, player 1 gets the I dollars, and gets the B amount back. If player 2 bets, he has to pay B dollars to the common fund, and the game goes to the next stage. (v) Both players show their card; the player with the highest number wins the amount in the common fund, that is the I amount and the amount $2 \times B$ paid in the earlier stages.

The utility of monetary amounts is linear. The equilibria may depend on the values of the parameters I and B : please specify the equilibrium set as a function of these parameters.

(b) Find the Nash equilibria of the game described in the first part, where the two players pick each one out of four cards, numbered $1, 2, 3,$ and 4. In the final stage card 4 beats card 3, which beats card 2, which beats card 1.

95 Rahman

Question IV.1 Fall 2009 majors

A 0 -normalized 2 -person bargaining game is a subset $S \subset \mathbb{R}_+^2$ that satisfies the following three conditions:

(a) S is convex and compact.

(b) S is comprehensive; i.e., if $x \in S, y \in \mathbb{R}_+^2$, and $y \leq x$, then $y \in S$

(c) There is an $x \in S$ such that $x_i > 0$ for $i = 1, 2$ Let β denote the set of all games. A bargaining solution is a function $\mu : \beta \to R_+^2$ such that $\mu(S) \in S$ for every $S \in \beta$ Consider the following four axioms: wask Paredo Optimality There is no $x \in S$ such that $x_i > (\mu(S))$, for $i = 1, 2$ $\cdot$ Homogeneity: $\mu(cS) = c\mu(S)$ for every $c > 0$ $\cdot$ Strong Individual Rationality: $(\mu(S))_i > 0$ for $i = 1, 2$ $\cdot$ Monotonicity: If $S \subseteq T$, then $(\mu(S))_i \leq (\mu(T))_i$ for $i = 1, 2$ We say that a solution μ is proportional if there are strictly positive constants, p_1 and p_2, such that for every $S \in \beta$, we have $\mu(S) = \lambda(S)p$, where $p = (p_1, p_2)$, and

$$\lambda(S) = \max\{t : \text{tp} \in S\}$$

Prove that a solution is weakly Pareto optimal, homogeneous, strongly individually rational, and monotonic if and only if it is proportional. Hint: Let μ be any solution that satisfies the four axioms. Let Δ be the triangle with vertices $(0,0), (0,1)$, and $(1,0)$. Define p to be the vector $\mu(\Delta)$

96 Rahman

Question IV.2 Fall 2009 majors

Consider an independent private-value auction with two bidders and one indivisible object. Both bidders have quasi-linear preferences, with each bidder i 's valuation, v_i, being independently and uniformly distributed between 0 and 1

Consider the following allocation rule: bidder 2 gets the object when $v_2 > v_1 + 1/2$ and bidder 1 gets the object otherwise. In the following questions, "mechanism" can be either direct or indirect, and "implementation" means truthful implementation.

(a) Construct a mechanism that implements the above allocation rule in dominant strategy equilibrium.

(b) Prove that there is no budget-balanced mechanism that can implement the above allocation rule in dominant strategy equilibrium.

(c) Construct a budget-balanced mechanism that implements the above allocation rule in Bayesian Nash equilibrium.

97 Werner

Question I.1 Spring 2010 majors

Let $Y \subset R^L$ be a finite production set, i.e., a set consisting of a finite number of production plans, $Y = \{y^1, y^2, \ldots, y^n\}$. Let $\hat{Y}$ be the convex hull of Y, that is the set of all convex combinations of production plans $y^1, \ldots, y^n$. Further, let $\pi_Y^*, s_Y^*, \pi_{\hat{Y}}^*$ and $s_{\hat{Y}}^*$ be the (maximum) profit functions and the supply correspondences for Y and $\hat{Y}$, respectively.

(a) Show that $\pi_Y^*(p) = \pi_{\hat{Y}}^*(p)$ for every $p \in \mathbb{R}^L$

(b) Show that $s_Y^*(p) \subset s_{\hat{Y}}^*(p)$ for every $p \in \mathbb{R}^L$. Show that the inclusion is strict for some price vectors if $n \geq 2$

98 Werner

Question I.2 Spring 2010 majors

Suppose that uncertainty is described by S states of nature with $S \geq 3$. Consider the following preference relation $\succeq$ on the set of state contingent consumption plans (or acts) $\mathbb{R}_+^S$ $c \succeq c'$ if and only if $\min c_s \geq \min c'_s$

(a) Show that $\succeq$ does not have state-separable utility representation.

(b) Consider a probability vector $\pi = (\pi_1, \ldots, \pi_S)$ such that $\pi_s > 0$ for all s. Show that $\succeq$ is strictly risk averse with respect to π. Derive risk compensation for risky gamble $z \in \mathbb{R}^S$ at deterministic initial wealth w where $E_\pi(z) = 0$

99 Allen

Question II.1 Spring 2010 majors

Consider a two-person two-commodity pure exchange economy with no free disposal in which each consumer has consumption set $\bar{R}_+^2$ and initial endowment (1,1). Assume that the preferences of each consumer are arbitrary and can be different for the two consumers, but can be represented by utility functions

$$u_1 : \mathbb{R}_+^2 \to \mathbb{R} \quad \text{and}$$
$$u_2 : \mathbb{R}_+^2 \to \mathbb{R}$$

respectively.

(a) What conditions must be satisfied by the preference relations $\preccurlyeq_1$ and $\preccurlyeq_2$ of each consumer?

(b) For this economy, define competitive equilibrium using the notation specifed above.

(c) For this economy and using this notation, define the sets of weakly and strongly Pareto optimal allocations.

(d) Is it possible that all Pareto optimal allocations give each consumer a utility exactly equal to one while all other feasible allocations give each a utility of exactly zero? Provide an example or explain why there can be no such example.

(e) Is it possible to have such an economy in which, at any Pareto optimal allocation, consumer 1's utility is strictly greater than consumer 2's utility? Explain why or why not.

(f) Now let the initial endowment vectors be $e_1 \in \mathbb{R}_+^2$ and $e_2 \in \mathbb{R}_+^2$ but $e_1 + e_2 = (2, 2)$ Can there be an economy in which $u_1(x_1^*, y_1^*) > u_1(x_1, y_1)$ where (x_1^*, y_1^*) is any competitive equilibrium allocation for consumer 1 and (x_1, y_1) is any feasible allocation for 1 which is not a competitive equilibrium allocation. Explain. Can the analogous statement hold for consumer 2 's competitive equilibrium and feasible allocations with the inequality reversed (i.e., $u_2(x_2^*, y_2^*) < u_2(x_2, y_2)$)? Explain.

(g) What additional assumptions on preferences are needed for the Second Welfare Theorem?

100 Allen

Question II.2 Spring 2010 majors

This question asks you to compare and contrast results regarding the existence and characterization of competitive equilibria in pure exchange economies in which all (finitely many) traders, $i = 1, \ldots, n$ have complete continuous preorders $\ll_i$ as preference versus twice continuously differentiable utility functions u_i representing smooth preferences. Both economies have ℓ (finite integer) perfectly divisible commodities.

(a) For each model, define competitive equilibrium.

(b) For each economy, state conditions that guarantee existence of competitive equilibrium using the notation given above for each economy.

(c) Again using the notation given above for each economy, state the Second Fundamental Theorem of Welfare Economics.

(d) For each economy, what can be said about the set of competitive equilibrium prices? For this question, you should continue to make the assumptions you used in part (c) for each economy and you may make additional assumptions providing that you state them clearly, explain their economic significance, and explain why they help you to obtain sharper results here.

(e) Given the above, when should we use each model (if we should) in microeconomic theory? List and briefly explain considerations that would cause you to favor first a model with continuous preferences and second a model with twice continously differentiable utility functions.

101 Rustichini

Question III.1 Spring 2010 majors

Define normal form and agent normal form of an extensive form game. Define a perfect equilibrium of an extensive form game (this is also known as Selten's Trembling Hand equilibrium). Then:

(a) Prove that an equilibrium is a perfect equilibrium of the extensive form if and only if it is a perfect equilibrium of the agent normal form of the game.

(b) Show by an example that a perfect equilibrium of the extensive form does not induce a perfect equilibrium of the normal form.

(c) Show by an example that a perfect equilibrium of the normal form does not induce a perfect equilibrium of the extensive form.

102 Rustichini

Question III.2 Spring 2010 majors

Consider a finite normal form game. Define a correlated strategy and equilibrium. (a) Define formally the communication extension of the normal form game, where an action profile is chosen according to a probability which is known to all players, and then a signal is announced to players with a probability over signals dependent on the action profile. Two extensions may be defined: (i) the same signal has to be announced to all players or (ii) the signal may be different.

(b) Prove that the equilibria of one of the two extended games described in option (i) and (ii) of point 1 above correspond to the set of correlated equilibria. State clearly which of the two games you are considering.

(c) What corresponds to the equilibria of the other game?

103 Rahman

Question IV.1 Spring 2010 majors

State the four axioms of the Nash bargaining solution; and prove that the Nash bargaining solution is the only bargaining solution that satisfies these four axioms.

104 Rahman

Question IV.2 Spring 2010 majors

Suppose there are two states of nature: $\omega = 0$ and $\omega = 1$. Let x and $\hat{x}$ be two signals of ω. Signal x has I (I finite) possible realizations, and the probability of the i th realization conditional on state ω is p^i_ω. Signal $\hat{x}$ has J (J finite) possible realizations, and the probability of the j th realization conditional on state ω is $\hat{p}^j_\omega$. Let $l^i := p^i_0/p^i_1$ and $\hat{l}^j := \hat{p}^j_0/\hat{p}^j_1$. Let $F(y)$ be a discrete distribution such that $y = l^i$ with probability p^i_1, and $\hat{F}(y)$ is a discrete distribution such that $y = \hat{l}^3$ with probability $\hat{p}^j_1$
Suppose x is more Blackwell-informative than $\hat{x}$ as a signal of ω. Prove that F is a mean-preserving spread of $\hat{F}$.

105 Rahman

Question IV.3 Spring 2010 majors

Consider the following 2-period adverse-selection problem. The principal relies on an agent to produce multiple units of a good in each period. Her objective function is $V = S(q_1) - t_1 + \delta(S(q_2) - t_2)$, where q_i (respectively t_i) is output (respectively transfer) in period $i, \delta \in (0,1)$ is a common discount factor, and $S(q)$ is a gross profit function that satisfies $S' > 0, S'' < 0, S(0) = 0$, and $S'(0) = \infty$ The agent's total production cost is linear in output. Let θ_i be his marginal cost in period i. His objective function is $U = t_1 - \theta_1 q_1 + \delta(t_2 - \theta_2 q_2)$

The marginal costs are unobservable to the principal. It is commonly known that, in period $1, \theta_1 = \underline{\theta}$ (respectively $\theta_1 = \bar{\theta}$) with probability ν_1 (respectively $1 - \nu_1$); and in period 2, conditional on $\theta_1, \theta_2 = \underline{\theta}$ (respectively $\theta_2 = \bar{\theta}$) with probability $\nu_2(\theta_1)$ (respectively $1 - \nu_1(\theta_1)$). We assume negative correlation: $0 < \nu_2(\underline{\theta}) < \nu_2(\bar{\theta}) < 1$ The timing of contracting is as follows. First, θ_1 is drawn and is observable only to the agent. Second, the principal offers a long-term contract to the agent for both periods. The agent then either accepts or rejects the contract. If he rejects, his reservation utility is 0. If he accepts, he cannot quit the relationship after the first period. Marginal cost θ_2 is drawn at the beginning of the second period. What is the principal's optimal contract for different combinations of $\Delta v_2 \equiv \nu_2(\underline{\theta}) - \nu_2(\bar{\theta}) \in (-1,0)$ and $\delta \in (0,1)$?

106 Werner

Question I.1 Fall 2010 majors

Consider a firm with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$. The firm maximizes its profit at prices $w \in \mathbb{R}^n_{++}$ for inputs and $q \in \mathbb{R}_{++}$ for output. The firm's profit maximization problem is

$$\max_{x \geq 0} qf(x) - wx$$

Let $x^*(q,w)$ denote the solution (input demand) and assume that x^* is a (single-valued) function of q and w

(i) State a definition of production function f being supermodular and briefly explain the economic meaning of this definition. Is the production function of two inputs $f(x_1, x_2) = x_1^\alpha x_2^\beta$, where $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, supermodular? Justify your answer.

(ii) Show that if production function f (of n inputs) is supermodular, then firm's input demand x^* is monotone nondecreasing in q. Make any assumptions you like, but state those assumptions clearly.
You may use any well-known mathematical result provided that you state the result clearly and verify that its assumptions are satisfied.

107 Werner

Question I.2 Fall 2010 majors

Consider an agent with expected utility function $E[v(\cdot)]$, where the von Neumann-Morgenstern utility function v is strictly increasing.

(i) State a definition of risk compensation $\rho(w, \tilde{z})$ for risky gamble $\tilde{z}$ with $E(\tilde{z}) = 0$ at deterministic initial wealth w

Consider risk compensation $\rho(w, t\tilde{z})$ as a function of scale factor t for arbitrary $t \in \mathbb{R}_+$

(ii) Show that $\rho(w, t\tilde{z})$ is a strictly increasing function of t that takes zero value at $t = 0$ for every w and $\tilde{z}$ with $E(\tilde{z}) = 0$, if and only if the agent is strictly risk averse.

108 Allen

Question II.1 Fall 2010 majors

For this questions let $\mathcal{E}$ be a pure exchange economy with l commodities and n traders ($i = 1, \ldots, n$) each having consumption set R^{ℓ}_{+}, initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$, and preferences $\preccurlyeq$ on $\mathbb{R}^{\ell}_{+}$, which are assumed to be continuous complete preorders that are strictly monotone.

(a) Define competitive equilibrium in $\mathcal{E}$.

(b) True or false; clearly and precisely explain your answer. The economy $\mathcal{E}$ might not have a competitive equilibrium because its excess demand fails to be continuous, so that the usual fixed point argument cannot be correctly made.

(c) What is the relationship between competitive equilibrium allocations and Pareto optimal allocations in this economy? Precisely state any theorem(s) that apply and explain why the theorem/proof applies. If a standard result does not hold, clearly explain why not (which hypotheses fail and why the standard proof strategy cannot be correctly used in this situation) and give a simple counter-example.

109 Allen

Question II.2 Fall 2010 majors

Let $\mathcal{E}$ be a pure exchange economy (with no free disposal) with ℓ commodities and n consumers $i = 1, \ldots, n$, each having consumption set $\mathbb{R}^{\ell}_{+}$, initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$, and utility function $u_i : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ which is assumed to be continuous, strictly monotonically increasing, and strictly concave. The Negishi characterization says that the set S of Pareto optimal allocations can be expressed as

$$S = \Big\{ x \in \mathbb{R}^{\ell n}_{+} | \text{ there exists } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n}_{+} \text{ with } \textstyle\sum_{i=1}^{n} \lambda_i = 1 \text{ such that}$$
$$x \in \arg\max \Big\{ \textstyle\sum_{i=1}^{n} \lambda_i u_i(x_i) \, | x \in \mathbb{R}^{(n \text{and})}_{+} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i \Big\} \Big\}$$

(a) Use this to show that the set of Pareto optimal allocations is nonempty.

(b) Use the Maximum Theorem (state precisely the version that you use) and the Negishi characterization to prove that the set of Pareto optimal allocations is compact under the assumptions stated above.

(c) Suppose that utility functions change to $\bar{u}_i : \mathbb{R}^{\ell n}_{+} \to \mathbb{R}$ defined by $\bar{u}_i(x) = \sum_{i=1}^{n} u_i(x_i)/n$ for all $i = 1, \ldots, n$. Does this change the set of Pareto optimal allocations from the Pareto optimal set S? If so, how? Prove your answer. What can you say to characterize this set? Explain.

(d) Discuss the economic interpretation of the utility function $\bar{a}_i$ in part (c).

110 Rustichini

Question III.1 Fall 2010 majors

(a) Prove that a Subgame perfect equilibrium induces a subgame perfect equilibrium in every subgame of the original game.

(b) Consider a subgame G' of a game G, starting at a node x of G. Let β a profile of behavioral strategies, which is a Nash equilibrium of G'. Consider the truncated game obtained by replacing in G the subgame G' with a terminal node, and assign to this node the payoff induced by β, and let also γ be an equilibrium of such truncated game. Finally consider the behavioral strategy profile in G that is given by γ in all the nodes in G that are not in G', and by β in G'

Prove that this behavioral strategy profile is a Nash equilibrium of G.

111 Rustichini

Question III.2 Fall 2010 majors

(a) Define a normal form game and its mixed extension.

(b) Give an interpretation of payoffs in the mixed strategy extension.

(c) Define two games to be equivalent if they have the same set of players, the same action set for each player, and the same best response correspondence for every player. What is the class of transformations of the payoffs in a normal form game that give an equivalent game? Prove your answer.

112 Rahman

Question IV.1 Fall 2010 majors

Consider the following adverse selection problem. Both the principal and the agent are risk neutral. The principal relies on an agent to produce q units of a good, where $q \in \mathbb{R}_+$ is a contractible quantity. Her utility function is $V = S(q) - t$, where $t \in \mathbb{R}$ is monetary transfer from her to the agent, and $S(\cdot)$ is a gross profit function that satisfies $S' > 0, S'' < 0, S(0) = 0, \lim_{q \to 0} S'(q) = \infty$, and $\lim_{q \to \infty} S'(q) = 0$ The agent's total production cost is linear in q. Let θ be his marginal cost. His utilit function is $U = t - \theta q$. His reservation utility is 0
The marginal cost θ is the agent's private information, but is unobservable to the principal. The principal believes that $\theta = \underline{\theta}$ (respectively $\theta = \bar{\theta}$) with probability ν (respectively $1 - \nu$). We assume that $\bar{\theta} > \underline{\theta} > 0$ Suppose, on top of all these standard elements, the principal can also demand (in a contract) that the agent takes a specific test, and contingent the monetary transfer t on th test result. The test costs nothing, and the test result can be either "pass" or "fail". The probability that an agent passes the test depends on his θ, and is equal to $\pi(\theta)$

(a) Suppose the agent cannot decrease his probability of passing the test by faking failure, and suppose $\pi(\bar{\theta}) > \pi(\underline{\theta})$. What is the principal's optimal contract?

(b) Suppose the agent can artificially decrease his probability of passing the test by faking failure. In particular, suppose an agent with marginal cost θ can costlessly choose any passing probability in $[0, \pi(\theta)]$]. Suppose $\pi(\bar{\theta}) > \pi(\underline{\theta})$. What is the principal's optimal contract?

(c) Repeat parts (a) and (b) using the alternative assumption that $\pi(\bar{\theta}) < \pi(\underline{\theta})$

113 Rahman

Question IV.2 Fall 2010 majors

Consider the following 2-period moral hazard model. In the first period, the agent chooses a consumption level $c \in \mathbb{R}$. In the second period, he chooses an effort level $e \in \{0, 1\}$, with effort cost $\Psi(e)$, where $\Psi(0) = 0$ and $\Psi(1) = \Psi > 0$. Neither c nor e is observable to the principal. Output $q \in \{\underline{q}, \bar{q}\}$ is observed (and is verifiable) at the end of the second period. Probability that $q = \bar{q}$ is π_e, with $1 > \pi_1 > \pi_0 > 0$. Assume that the principal wants to induce effort $e = 1$, and tries to minimize the expected wage she pays the agent. If the wage contract is $t(q)$, and if the agent consumes c in the first period and exerts effort e in the second, then his utility is

$$u(c) + \pi_e u(t(\bar{q}) - c) + (1 - \pi_e)\, u(t(\underline{q}) - c) - \Psi(e)$$

where $u(\cdot)$ is strictly increasing and concave. The agent's reservation utility is 0.

(i) Assume that the principal can commit to any wage contract she offers to the agent at the beginning of the first period (i.e., before the agent chooses c). Set up the principal's problem (as a constrained optimization problem).

(ii) Let $t^*(q)$ be the solution to your problem in part (i). Let $\bar{c}$ be the agent's optimal choice of consumption if he is offered the contract $t^*(q)$ and if he is constrained to choose effort $e = 1$. Similarly, let $\underline{c}$ be the agent's optimal choice of consumption if he is offered the contract $t^*(q)$ and if he is constrained to choose effort $e = 0$. Prove that $\bar{c} > \underline{c}$

(iii) Use your result in part (ii) to prove that the contract $t^*(q)$ is not renegotiation-proof. In particular, suppose you were the principal, and suppose you had an opportunity to renegotiate with the agent at the end of the first period (i.e., right after he chose c), and suppose you're sure that he had just chosen $c = \bar{c}$. Explain how you can propose to modify the contract $t^*(q)$ to make both you and the agent strictly better off from that moment on.

114 Werner

Question I.1 Spring 2011 majors

Consider a utility function u on $\mathbb{R}^L_+$ defined by

$$u(x) = \inf_{q \in Q} qx$$

for every $x \in \mathbb{R}^L_+$. The set Q is a closed and convex subset of the unit simplex Δ in $\mathbb{R}^L$ and such that $Q \subset R^L_{++}$

(a) Show that utility function u is locally non-satiated and concave.

(b) Characterize the points x of differentiability of u. What is the derivative (or the gradient vector)$Du(x)$ at a point x of differentiability? You may use any well-known mathematical result pertaining to (b) without proof.

115 Werner

Question I.2 Spring 2011 majors

Let $\tilde{y}$ and $\tilde{z}$ be two random variables on some state space (i.e. probability space) such that $E(\tilde{z}) = 0$

(a) Give an example of $\tilde{y}$ and $\tilde{z}$ with $\tilde{z} \neq 0$ such that $\tilde{y} + \tilde{z}$ is more risky than $\tilde{y}$. Justify your claim.

(b) Give an example of $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y} + \tilde{z}$ is not more risky than $\tilde{y}$. Justify your claim.

(c) Show that if $\tilde{y} + \tilde{z}$ is more risky than $\tilde{y}$, then $\tilde{y} + 2\tilde{z}$ is more risky than $\tilde{y}$, too. Your definition of more risky should be stated in terms of cumulative distribution functions of random variables. You may use any well-known characterization of more risky without proof, but you need to state it clearly.

116 Allen

Question II.1 Spring 2011 majors

Consider a pure exchange economy with ℓ commodities and n consumers, each having consumption set $\mathbb{R}^{\ell}_{+}$, initial endowment vector $e_i \in \mathbb{R}^{\ell}_{++}$, and preferences $\preccurlyeq_i$ represented by a utility function $u_i : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ which is assumed to be continuous and strictly monotone. Now consider the following statement about such economies: There might not exist a competitive equilibrium because the potential lack of at least quasi-concavity of utilities prevents one from applying the Maximum Theorem to obtain continuity of (individual and hence aggregate) excess demand, so that Kakutani's Fixed Point Theorem cannot be used.

(a) Is this statement true or false? Pick one and draw a box around your answer. Explain your reasoning very carefully.

(b) Give an example of such an economy with $\ell = n = 2$ that does not have a competitive equilibrium.

(c) State the Maximum Theorem.

(d) State Kakutani's Fixed Point Theorem and define a fixed point of a correspondence.

(e) What assumptions must the preferences $\preccurlyeq_i$ in this economy satisfy?

(f) Without reference to the utilities u_i, define competitive equilibrium for this economy.

117 Allen

Question II.2 Spring 2011 majors

This question asks you to compare and contrast different alternative approaches to general equilibrium models of pure exchange economies. Throughout please use the notation that there are (where ℓ and n are finite positive integers greater than two) ℓ commodities and n agents, indicated by subscripts $i = 1, \ldots, n$, each with initial endowment vector $e_i \in \mathbb{R}^\ell_{++}$ Economy A, denoted $\mathcal{E}(A)$ has preferences specified by continuous complete preorders $<$ in an $\mathbb{R}^\ell_+$ which are strictly monotone and at least weakly convex. Economy B, denoted $\mathcal{E}(B)$, has preferences as in $\mathcal{E}(A)$ represented by continuous utility functions $u_i : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfy further assumptions (to be specified by you) such that demands are continuous functions on $\Delta = \left\{p \in \mathbb{R}^\ell_{++} | \sum_{j=1}^\ell p_j = 1\right\}$. Economy C, denoted $\mathcal{E}(C)$, is smooth and satisfies further assumptions on the u_i such that demands are (at least once) continuously differentiable functions on Δ.

(a) Define continuity of a complete preorder $\preccurlyeq_i$ on $\mathbb{R}^\ell_+$. Define weak convexity of a continuous complete preorder $\preccurlyeq_i$ on $\mathbb{R}^\ell_+$.

(b) What conditions do the utilities u_i representing $\ll i$ as in $\mathcal{E}(A)$ necessarily satisfy? What additional assumption(s) is/are needed to guarantee that demands are continuous functions for economy $\mathcal{E}(B)$?

(c) State all additional assumptions that are needed for $\mathcal{E}(C)$ to have continuously differentiable demand functions.

(d) What can we say about the set of (normalized) equilibrium prices in Δ for each of these economies, $\mathcal{E}(A), \mathcal{E}(B)$, and $\mathcal{E}(C)$? Specifically, is the set nonempty and if so, how large can it be? What tools (theorems) are needed for your argument? Please explain your reasoning but do not write a proof.

(e) What are the advantages and disadvantages of using each model $\mathcal{E}(A), \mathcal{E}(B)$, and $\mathcal{E}(C)$ in microeconomic theory? When and why should we choose one of these models over the other?

118 Rustichini

Question III.1 Spring 2011 majors

(a) Find all the Nash equilibria in the following game:

	l	m
T	$1, 10$	$0, 0$
M	$1, 2$	$1, 2$
B	$3, -10$	$0, 0$

(b) Find all the perfect equilibria of the same game

(c) Take a Nash equilibrium $\hat{\sigma}$ of any normal form finite game, and remove the action of a player (i) which is not a best response to the strategy of the others, σ^{-i}. Is the restriction of the strategy profile $\hat{\sigma}$ to the new game a Nash equilibrium of the new game? Prove your answer.

(d) Take a perfect equilibrium $\hat{\sigma}$ of any normal form finite game, and remove the action of a player (i) which is not a best response to the strategy of the others, σ^{-i}. Is the restriction of the strategy profile $\hat{\sigma}$ to the new game a perfect equilibrium of the new game? Prove your answer.

119 Rustichini

Question III.2 Spring 2011 majors

Consider a normal form game with a set I of players. Part 1 Show that the set of strategies surviving iterated elimination of weakly dominated strategies is non-empty Part 2 A Strong Nash Equilibrium is a strategy profile $(\sigma_i)_{i\in I}$ such that for no subset J of players there exists a strategy profile $(\tau_j)_{j\in J}$ that gives to each player in the set J a higher payoff than the profile $(\ \sigma_j\ )\ j \in J$, while every player $k \in I\backslash J$ plays the strategy σ_k.

(a) Show an example of a Strong Nash Equilibrium in a game with two players.

(b) Show an example of a Strong Nash Equilibrium in a game with three or more players.

(c) Does a Strong Nash equilibrium always exist? Prove or disprove with an example.

120 Rustichini

Question IV.1 Spring 2011 majors

Below you will find several normal form games; these are exactly those for which you found the Nash equilibria in the past. For each of these games find:

(a) The set of Nash equilibria and Nash equilibrium payoffs for the stage game

(b) The set $co\left(F^0\right)$, where $F^0 \equiv \{u(a) : a \in A\}$, and A is the set of action profiles

(c) The vector of minimax values $(\ v^i\ )_{\ i\in I}$

(d) The set of feasible and individually rational payoffs

The games (a)

	l	r
T	$2,2$	$-1,-1$
B	$0,4$	$0,4$

(b)

	l	m	r
T	$-1,8$	$2,5$	$4,3$
B	$1,2$	$2,4$	$1,9$

(c)

	l	m	r
T	1,1	0,0	0,0
M	0,0	4,4	8,0
B	0,0	0,10	9,9

121 Rustichini

Question IV.2 Spring 2011 majors

Consider a repeated game with imperfect monitoring (the action chosen by the players is not observed) and public monitoring (a public signal, like the quantity produced or the price in an oligopoly model is observed). Define a perfect equilibrium in public strategies. Prove that the equilibrium corresponds to a subgame perfect equilibrium of the repeated game where players may use private information.

122 Rahman

Question IV.3 Spring 2011 majors

Consider the following moral hazard problem. There are two effort levels: $e = 0$ and $e = 1$. The agent is an expected-utility maximizer. If he puts in effort e, and subsequently receives monetary payment t, then his utility will be

$$u(e,t) = \begin{cases} t & \text{if } e = 0 \\ \frac{1}{2} - \exp(-t) & \text{if } e = 1 \end{cases}$$

Notice that $u(0,t) > u(1,t)$ for all t; and that while $u(0,\cdot)$ is linear, $u(1,\cdot)$ is concave. The agent's reservation utility is 0.
The principal's utility is $v(e,t) = e - t$ if the agent puts in effort e and she pays the agent monetary payment t

(a) Suppose the agent's effort is neither observable (to the principal) nor verifiable (to the court). What is the cheapest way for the principal to induce the agent to put in effort $e = 1$?

(b) Suppose the agent's effort is observable (to the principal) but not verifiable (to the court). What is the cheapest way for the principal to induce the agent to put in effort $e = 1$?

123 Werner

Question I.1 Fall 2011 majors

Consider an agent facing uncertainty described by a finite set of states S. The agent's preferences over state-contingent consumption plans $x \in \mathbb{R}^S_+$ are described by the utility function

$$u(x) = \inf_{\pi \in P} E_\pi x$$

where $E_\pi x = \sum_{n=1}^S \pi_s x_s$ denotes the expected value of x and $\mathcal{P}$ is a set of probability measures on S. Set $\mathcal{P}$ is a subset of the unit simplex Δ in R^S and is assumed closed, convex, and such that $\mathcal{P} \subset \mathcal{DR}^S_{++}$

(a) Show that utility function u is locally non-satiated and concave.

(b) Characterize the points x of differentiability of u. What is the derivative (or the gradient vector)$Du(x)$ at a point x of differentiability?

You may use any well-known mathematical result pertaining to (b) without proof.

124 Werner

Question I.2 Fall 2011 majors

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values.

(a) State a definition of $\tilde{z}$ first-order stochastically dominating (FSD) $\tilde{y}$. Show that if $\tilde{z}$ FSD $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$

(b) Show that, if $\tilde{z}$ FSD $\tilde{y}$ and $E(\tilde{z}) = E(\tilde{y})$, then $\tilde{y}$ and $\tilde{z}$ have the same distribution, i.e., $F_y(t) = F_z(t)$ for every $t \in \mathbb{R}$. If you find it convenient, you may assume in your proof that random variables $\tilde{y}$ and $\tilde{z}$ have densities, or alternatively that $\tilde{y}$ and $\tilde{z}$ are discrete random variables (i.e., take finitely many values). (c) State a definition of $\tilde{z}$ second-order stochastically dominating (SSD) $\tilde{y}$. Show that if $\tilde{z}$ ssn $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$

(d) Show that if $\tilde{z}$ FSD $\tilde{y}$, then $\tilde{z}$ SSD $\tilde{y}$.

(e) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.

125 Allen

Question II.1 Fall 2011 majors

Consider a pure exchange economy with n consumers $i = 1, \ldots, n$, each having initial endowment $e_i \in \mathbb{R}_+^\ell$ and preferences $\preceq_i$ assumed to be complete preorders on $\mathbb{R}_+^\ell$. (a) Define the core of this economy.

(b) Define the m-replica economy for any integer $m \geq 1$. State and prove the Equal Treatment Property for the core of the m -replica economy.

(c) State the Debreu-Scarf Theorem and discuss its economic significance.

126 Allen

Question II.2 Fall 2011 majors

Let $\mathcal{E}$ be a pure exchange economy (with no free disposal) with ℓ commodities and n consumers $i = 1, \ldots, n$, each having consumption set $\mathbb{R}_+^\ell$, initial endowment vector $e_i \in \mathbb{R}_{++}^\ell$, and utility function $u_i : \mathbb{R}_+^\ell \to \mathbb{R}$ which is assumed to be continuous, strictly monotonically increasing, and strictly concave. Let S be the set of Pareto optimal allocations of $\mathcal{E}$

(a) Consider an allocation $\bar{x}$ that solves the following maximization problem

$$\begin{array}{cl} \max_x & \sum_{i=1}^n \lambda_i u_i(x_i) \\ \text{subject to} & \sum_{i=1}^n x_i = \sum_{i=1}^n e_i \\ & x_i \in \mathbb{R}_+^l, \forall i \end{array}$$

for some $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda \geq 0$ and $\lambda \neq 0$. Show that allocation $\bar{x}$ is Pareto optimal.

(b) Prove that the set S of Pareto optimal allocations is non-empty.

(c) Prove that the set S is compact.

127 Rustichini

Question III.1 Fall 2011 majors

In a finite game with $\{1, \ldots, n\}$ players let A^i be the action set of player i, and A the set of action profiles. A finite automaton (FA) for player i is $M^i \equiv \left(X^i, \tau^i, f^i, x_1^i\right)$ where X^i is a finite set, $\tau^i : X^i \times A \to X^i$ the transition function on $X^i, f^i : X^i \to A^i$, and $x_1^i \in X^i$ the initial state.

(a) Prove that an FA induces a unique pure strategy in an infinitely repeated game

(b) What can you say about the converse? Prove your answer, providing a counterexample if appropriate.

128 Rustichini

Question III.2 Fall 2011 majors

(a) Define Backward Induction for a finite extensive form game.

(b) Prove that Backward Induction always produces a non empty set of strategy profiles. Is an element in this set of strategy profiles a Subgame Perfect Equilibrium? Is it a Perfect Equilibrium?

(c) Give an example of a finite game of perfect information in which the Backward Induction strategies are not unique, but the payoff vector is unique.

129 Rustichini

Question IV.1 Fall 2011 majors

Consider the game which is obtained as a finite repetition for N periods of the Prisoner's dilemma stage game

	l	r
T	$2, 2$	$-1, 3$
B	$3, -1$	$0, 0$

for a finite number of periods, and with final payoff equal to the sum on the payoffs in every period

(a) Find the set of the Subgame perfect equilibria

(b) Find the set of Nash equilibria. Prove your answer.

(c) Now note that the PD is a game with the following property (M): The vector of minimax values is equal to the unique Nash equilibrium payoff. Consider any two players stage game with finite action set and with property M. Find Subgame perfect and Nash equilibria for any such game.

130 Rustichini

Question IV.2 Fall 2011 majors

Consider the infinitely repeated game, discounted by $\delta \in (0, 1)$, where the stage game

is Matching pennies game:

	l	r
T	$2, -2$	$-2, 2$
B	$-2, 2$	$2, -2$

(a) Describe the set of behavioral strategy Nash equilibria of the repeated game.

(b) Describe the set of mixed strategy Nash equilibria of the repeated game.

131 Rahman

Question IV.3 Fall 2011 majors

Consider the following moral hazard problem. The principal has a business project, but is unsure about its quality, q. With probability $\gamma \in (0,1)$, the project has high quality ($q = h$), and the principal will make a positive profit of $y_h > 0$ if she pursues it. With probability $1-\gamma$, the project has low quality ($q = l$), and the principal will make a negative profit of $y_l < 0$ if she pursues it.

The agent is an expert who provides the principal with information about the project's quality. He can observe a signal, $\theta \in \{\theta_h, \theta_l\}$, that is correlated with q. How informative the signal is depends on how much effort, $e \geq 0$, he exerts. In particular,

$$\Pr\{\theta = \theta_h | q = h, e\} = \alpha + \beta_h \eta(e) \text{ and } \Pr\{\theta = \theta_h | q = l, e\} = \alpha - \beta_l \eta(e)$$

where $\alpha \in (0,1), \beta_h, \beta_l > 0$, and the function η satisfies $\eta(0) = 0, \eta'(e) > 0, \eta''(e) < 0$ for all e, and $\lim_{e \to \infty} \eta(e) = \eta < \min\{(1-\alpha)/\beta_h, \alpha/\beta_l\}$ Exerting effort, e, is costly for the agent, which entails a disutility of $\psi(e)$, where the function ψ satisfies $\psi(0) = 0, \psi'(0) = 0, \psi'(e) \geq 0$, and $\psi''(e) > 0$ for all e The agent's effort is unobservable, but he cannot lie about the signal realization. Therefore, a contract is a pair of numbers, (w_h, w_l), specifying the wages paid to the agent if the signal realization is θ_h and θ_l, respectively.

Both the principal and the agent are risk neutral. The principal's payoff equals to her expected profit minus the expected wage paid to the agent. The agent's payoff equals to the expected wage received minus the disutility of effort.

The agent's reservation utility is 0. He is protected by limited liability, and thus wages are restricted to be nonnegative. (a) Set up the optimal contract design problem. (You do not need to solve it.) (b) Suppose the optimal contract induces a strictly positive effort level (i.e., $e^* > 0$). Show that, if $\gamma < \beta_i / (\beta_h + \beta_l)$, then the optimal contract must have the property that $w_i^* > 0 = w_h^*$; that is, the agent gets paid only when he tells the principal that her project has low quality. Explain intuitively why the principal seems to be more pleased with bad news in this case. (c) What is the optimal contract if $\gamma = \beta_l / (\beta_h + \beta_l)$?

132 Werner

Question I.1 Spring 2012 majors

Let $u : \mathbb{R}_+^L \to \mathbb{R}$ be a continuous and strictly increasing utility function. Let $u^*(p, M)$ be the indirect utility function derived from the maximization of utility u subject to the budget constraint at prices $p \in \mathbb{R}_{++}^L$ and income $M > 0$

(a) Show that if u is concave, then u^* is a concave function of income M

(b) Show that if u is strictly concave, then Walrasian demand $x^*(p, M)$ is single-valued for every (p, M)

(c) Suppose that u is quasi-linear of the form

$$u(x) = x_1 + v(x_2, \ldots, x_L)$$

for every $x \in \mathbb{R}_+^L$. Function $v : \mathbb{R}_+^{L-1} \to \mathbb{R}$ is assumed continuous, strictly increasing, and strictly concave. Show that indirect utility u^* is linear in M on the domain of price-income pairs (p, M) where the Walrasian demand is interior. If your proof relies on differentiability of the demand function and/or the indirect utility function, you should clearly state additional assumptions pertaining to differentiability of function v that are sufficient for these desired properties of demand and indirect utility, and justify sufficiency of your assumptions.

133 Werner

Question I.2 Spring 2012 majors

Consider an agent whose preferences over state-contingent consumption plans $c \in R_+^S$ have state-separable representation of the form

$$U(c) = \sum_{s=1}^{S} u_s\left(c_s\right)$$

where S is a finite number of states of nature. Functions $u_s : \mathbb{R}_+ \to \mathbb{R}$ are assumed strictly increasing and differentiable for every s. Let π be a probability measure on states such that $\pi_s > 0$ for every s

(a) Show that if the agent is risk averse with respect to probability measure π, then utility function U must have an expected utility representation under π. That is, the preference relation induced by U on $\mathbb{R}_+^S$ must have and expected utility representation.

(b) Show that the von Neumann-Morgenstern utility function of the expected utility representation in part (a) must be concave. [You may use the Theorem of Pratt in your proof, but you need to state it clearly and explain how it implies the claim.]

134 Allen

Question II.1 Spring 2012 majors

It is frequently stated that convexity matters for one of the welfare theorems but not the other. This question asks you to explore this claim in more detail, in the context of a pure exchange economy with ℓ commodities and n consumers (indexed by $i = 1, 2, \ldots, n$), each having preferences $\preccurlyeq_i$ defined on $\mathbb{R}_+^\ell$ which are assumed throughout to be continuous complete preorders, and initial endowment vectors $e_i \in \mathbb{R}_+^\ell$.

(a) For a complete preorder $\preccurlyeq_i$, define continuity.

(b) State the Second Welfare Theorem.

(c) State the First Welfare Theorem and give a proof. Now suppose for the remainder of this question that consumers' feasible consumption sets are $X_i \subseteq \mathbb{R}_+^\ell$

(d) Does your statement of the Second Welfare Theorem require any additional assumptions? If so, state them and explain briefly why they are needed.

(e) Does your statement of the First Welfare Theorem require any additional assumptions? If so, state them, explain intuitively why they are needed, and indicate where the new assumptions would be used in your proof in part (c).

(f) Discuss. Is it true that one of the welfare theorems does not require convexity?

135 Allen

Question II.2 Spring 2012 majors

Local television news media have recently reported that Delta Air Lines has been offering tickets on the same flights and same days at different prices to different customers, in particular showing higher prices sometimes to people that Delta knows have frequent flyer accounts and travel often (because this happened to people who are logged into their accounts when purchasing tickets and fly at least $50,000$ miles per year credited to Delta). This question asks you to think about how one might analyze this in a general equilibrium model.

(a) How would you specify any systematic differences (what might be such differences, if any?) between frequent flyers and other consumers in your model?

(b) How would you model the idea of different prices for different customers formally? Be precise about the economic environment and any relevant assumptions in a formal general equilibrium model. To simplify, you may decide to neglect production and use a pure exchange economy since in the short run, Delta has already decided what products (routes, seats, meals, etc.) to produce and how much to produce.

(c) Discuss the implications for the existence of competitive equilibrium and both welfare theorems. Identify precisely which, if any, of the standard assumptions would not be valid in this situation. Define equilibrium and Pareto optimal allocations.

(d) Now suppose that instead of systematic price differences based on frequent flyer status, Delta is simply experimenting with pure randomized prices. How would you model this?

(e) Again for the case of random prices not correlated with frequent flyer status, what would be the implications for the existence of equilibrium and the welfare theorems? Define equilibrium and Pareto optimality in your basic model.

136 Rustichini

Question III.1 Spring 2012 majors

(a) Define a sequential equilibrium of an extensive form game.

(b) Provide a proof of existence of sequential equilibria. A sketch will illustrate the main idea, and explain which the main technical difficulties would be and how you would deal with those.

(c) Show that for finite extensive form games, if a behavioral strategy τ is completely mixed, then

- Every information set is reached with positive probability;
- The assessment (μ, τ) is consistent if and only if μ can be derived by Bayes' rule.
- Is every sequential equilibrium trembling hand perfect? Prove or disprove.

137 Rustichini

Question III.2 Spring 2012 majors

(a) Define Iterated Elimination of Strictly Dominated Strategies (IESDS), and Iterated Elimination of Weakly Dominated Strategies (IEWDS);

(b) Prove or disprove (with an example): all IEWDS procedures yields a unique outcome

(c) Prove or disprove (with an example): IESDS cannot eliminate a Nash Equilibrium;

(d) Prove or disprove (with an example): IEWDS cannot eliminate a Nash Equilibrium.

138 Rahman

Question IV.1 Spring 2012 majors

An auctioneer is auctioning an indivisible object among two symmetric bidders. Both bidders have quasilinear preference, and each bidder i 's valuation of the object, t_i, is drawn independently from a uniform distribution over $[0,1]$, and is known only to bidder i himself. The auctioneer considers using one of the following two auctions:

Auction 1: Bidders simultaneously submit nonnegative bids. The bidder who submits the highest bid wins the object. If both bidders submit the same bid, winner is determined by a fair coin. A bidder bidding $b \geq 0$ has to pay b if he wins, and has to pay b^2 even if he loses.

Auction 2: Bidders simultaneously submit nonnegative bids. The bidder who submits the highest bid wins the object. If both bidders submit the same bid, winner is determined by a fair coin. A bidder bidding $b \geq 0$ has to pay b if he wins, and will receive b from the auctioneer if he loses.

For each of these auctions, determine whether there is an efficient Bayesian Nash equilibrium (BNE) (that is, a BNE where the winner of the object is always a bidder with the highest valuation). If your answer is yes, explicitly construct one (you have to show carefully that it indeed is an efficient BNE), and calculate the auctioneer's expected revenue in that equilibrium. If your answer is no, explain why this is so. If your answer is yes for both auctions, does the auctioneer collect the same expected revenue from the two auctions?

139 Rahman

Question IV.2 Spring 2012 majors

Let Γ be a finite set of alternatives, with $|\Gamma| \geq 3$. Let E be a finite set of individuals, with $|E| = n$. Individual preferences are represented by a binary relation on Γ which is complete, anti-symmetric, and transitive. Let P_i denote such a binary relation for individual i, and xP_iy means i prefers x to y. Let Σ denote the set of all such binary relations on Γ. For a given P_i, let xR_iy mean either xP_iy or $x = y$.

A profile of preferences is a vector $\mathbf{P} = (P_1, \ldots, P_n) \in \Sigma^n$. A social choice function (SCF) is a mapping $f : \Sigma^n \to \Gamma$. An SCF is dictatorial if there exists $i \in E$ such that for any $x \in \Gamma$ and any $\mathbf{P} \in \Sigma^n$, $f(\mathbf{P})R_ix$.
A mechanism is a pair $M = (\mathbf{S}, g)$ where $\mathbf{S} = S_1 \times \cdots \times S_n$ and $g : \mathbf{S} \to \Gamma$. For any mechanism M and any preference profile $\mathbf{P} \in \Sigma^n$, the pair $G = (M, \mathbf{P})$ constitutes a complete-information game. For a given $G = (M, \mathbf{P})$, a strong equilibrium is a profile of actions $s \in \mathbf{S}$ such that for any coalition $C \subset E$ and any $\hat{\mathbf{s}}_C = (\hat{s}_i)_{i\in C} \in \mathbf{x}_{i\in C}S_i$, there exists $i \in C$ such that $g(\mathbf{s})R_ig\left(\hat{\mathbf{s}}_C, \mathbf{s}_{-C}\right)$, where $\mathbf{s}_{-C} = (s_j)_{j\notin C}$
Let $SE(G)$ denote the set of all strong equilibria of G. An SCF f is fully implementable in strong equilibrium if there exists $M = (\mathbf{S}, g)$ such that for any $\mathbf{P} \in \Sigma^n$ $g(SE(M, \mathbf{P})) = \{f(\mathbf{P})\}$
Prove that, if an SCF f is such that $f\left(\Sigma^n\right) = \Gamma$, then f is fully implementable in strong equilibrium only if it is dictatorial.

140 Allen

Question I.1 Fall 2012 majors

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}^n_+$ in an economy with two agents:

$$\max_x \mu u^1(x) + (1-\mu)u^2(\omega - x)$$
$$\text{subject to } x \leq \omega, x \geq 0$$

where $u^i : \mathbb{R}^n_+ \to \mathbb{R}$, for $i = 1, 2$, are agents' utility functions, and μ is the welfare weight of agent 1. μ lies in the interval [0,1]. Let $x^*(\mu)$ be a solution (assumed unique).

(a) State a definition of utility function u^i being supermodular in x. Give an example of a supermodular utility function other than the linear function. Justify your answer.

(b) Show that, if utility functions u^1 and u^2 are strictly increasing and supermodular in x, then $x^*(\mu)$ is non-decreasing in μ. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

141 Werner

Question I.2 Fall 2012 majors

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some probability space. Random variables $\tilde{y}$ and $\tilde{z}$ have the same expectations, $E(\tilde{y}) = E(\tilde{z})$

(a) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Show that, if $\tilde{y}$ is more risky than $\tilde{z}$ then $\text{var}(\tilde{y}) \geq \text{var}(\tilde{z})$, where var( ) denotes the variance.

(b) Give an example of two random variables $\tilde{y}$ and $\tilde{z}$ (with the same expectations) such that $\text{var}(\tilde{y}) > \text{var}(\tilde{z})$, but $\tilde{y}$ is not more risky than $\tilde{z}$. Justify your answer.

142 Allen

Question II.1 Fall 2012 majors

Local television news media have recently reported that Delta Air Lines has been offering tickets on the same flights and same days at different prices to different customers, in particular showing higher prices sometimes to people that Delta knows have frequent flyer accounts and travel often (because this happened to people who are logged into their accounts when purchasing tickets and fly at least $50,000$ miles per year credited to Delta). This question asks you to think about how one might analyze this in a general equilibrium model.

(a) How would you specify any systematic differences (what might be such differences, if any?) between frequent flyers and other consumers in your model?

(b) How would you model the idea of different prices for different customers formally? Be precise about the economic environment and any relevant assumptions in a formal general equilibrium model. To simplify, you may decide to neglect production and use a pure exchange economy since in the short run, Delta has already decided what products (routes, seats, meals, etc.) to produce and how much to produce.

(c) Discuss the implications for the existence of competitive equilibrium and both welfare theorems. Identify precisely which, if any, of the standard assumptions would not be valid in this situation. Define equilibrium and Pareto optimal allocations.

(d) Now suppose that instead of systematic price differences based on frequent flyer status, Delta is simply experimenting with pure randomized prices. How would you model this?

(e) Again for the case of random prices not correlated with frequent flyer status, what would be the implications for the existence of equilibrium and the welfare theorems? Define equilibrium and Pareto optimality in your basic model.

143 Allen

Question II.2 Fall 2012 majors

Consider a pure exchange economy with two commodities and three traders (indexed by subscripts $1, 2$, and 3), each having the same consumption sets $\mathbb{R}^2_+$ and the same initial endowment vector $e_1 = e_2 = e_3 = (1, 1)$. For $c \in \mathbb{R}_+$ (so that the scalar $c \geqq 0$ is a nonegative constant), the first agent's indifference curves are given by $\{(x, y) \in \mathbb{R}^2_+ | x = c \text{ and } y \geqq c \text{ or } x \geqq c \text{ and } y = c\}$ while the indifference curves of agents 2 and 3 are given by $\{(x, y) \in \mathbb{R}^2_+ | x + y = c\}$. For all agents, it should be understood that larger values of c correspond to higher (better) indifference curves.

(a) Are the preferences of these traders monotone? Strictly monotone? Convex? Strictly convex?

(b) Find a utility representation for each trader's preferences.

(c) Define competitive equilibrium in this economy. (Be sure to define any notation that you introduce.)

(d) Find all competitive equilibrium allocations in this economy.

(e) Find all competitive equilibrium price vectors in this economy.

(f) State a theorem on the existence of competitive equilibrum and explain why it does or does not apply to this economy. (I.e., does this economy satisfy the assumptions for the theorem you stated? Why or why not?)

(g) *SKETCH* a proof of the theorem you stated in part (f), including giving complete statements of any major mathematics theorem(s) that are key to the proof. DO NOT GIVE A COMPLETE PROOF - it will NOT receive credit here.

(h) Can you modify the demand relation of the first agent so that the first agent's excess demand correspondence does not cause the economy to fail to satisfy the assumptions of the Very Easy Existence Theorem? In other words, you are being asked how the Very Easy Existence Theorem can be applied - by modifying the demand relation of each agent in an innocuous way - to guarantee that an economy consisting (only) of N identical copies of agent 1 has a competitive equilibrium. (Ignore whether adding the excess demands of agents 2 and 3 to obtain aggregate excess demand for this economy causes any assumptions to fail to be satisfied.) Explain your answer. What does this indicate about the first agent's potential competitive equilibrium allocations in this economy? Explain.

144 Rustichini

Question III.1 Fall 2012 majors

A play in an extensive form game (EFG) is a complete history, from the initial node to one of the final nodes. An EFG is said to be linear if every information set is crossed only once in every play.

(a) Give an example of an EFG which is not linear.

(b) Compare linear games and games with perfect recall. Which set is a subset of the other? Prove your answer.

145 Rustichini

Question III.2 Fall 2012 majors

(a) Prove that the strategy profile inducing a sequential equilibrium is a sub-game perfect equilibrium.

(b) Are the two set of equilibria different? Show with an example that they are or prove that they are not.

146 Rustichini

Question IV.1 Fall 2012 majors

A seller is designing a selling mechanism to sell an indivisible object to a buyer. The seller has no use of the object, and aims at maximizing expected revenue. The buyer has quasilinear preference, whose valuation of the object, t, is known only to buyer himself. The seller knows only that t is either 2 or 3, with equal probabilities. It is commonly known that t is correlated with a random variable $\tilde{\tau} \in \{\tau, \tau'\}$. Conditional on $t = 3$, the probability that $\tilde{\tau} = \tau$ is $1/4$. Conditional on $t = 2$, the probability that $\tilde{\tau} = \tau$ is π. While the seller and the buyer agree that $\pi > 1/4$, they disagree on the exact value of π. While the seller thinks that $\pi = 3/4$, the buyer thinks that $\pi = 1/2$, and these heterogeneous beliefs are common knowledge among them. Since reasonable people can agree to disagree on probability assessments, the seller maintains her own belief even though she knows perfectly well that the buyer's belief is different from hers. What is the seller's optimal selling mechanism?

147 Rahman

Question IV.2 Fall 2012 majors

An auctioneer is designing an auction mechanism to auction an indivisible object among two bidders (1 and 2). The auctioneer has no use of the object, and aims at maximizing expected revenue. The bidders have quasilinear preferences, with their valuations of the object denoted by $t_i, i = 1, 2$. The values of $t_i, i = 1, 2$, are drawn independently from the uniform distribution over $[0, 1]$. Bidder 1 knows his valuation t_1 before deciding whether to participate the auction, whereas bidder 2 will learn his valuation t_2 only after he participates. The value of t_i will remain bidder i 's private information afterward. What is the auctioneer's optimal auction mechanism?

148 Rustichini

Question IV.3 Fall 2012 majors

Consider a repeated game with discounting given by δ.

(a) Define the operator SP^6, giving for every set of feasible payoffs the continuation values in subgame perfect equilibria

(b) Prove that the set of sub-game perfect equilibrium payoffs is a fixed point of SP^5

149 Werner

Question I.1 Spring 2013 majors

State the weak axiom of cost minimization for a finite set of observations of input (or factor) choices, input prices, and output quantities of a firm that produces single output good using n input goods in competitive markets.

(a) Show that the axiom you stated is necessary for rationalizability (the term you'll need to define clearly) of the observations by a production function.

(b) Consider a factor demand function ψ that assigns a vector of input choices $x = \psi(w, z)$ to every vector of strictly positive input prices w and every positive output quantity z. Assume that function ψ is differentiable function of prices w Show that if the weak axiom of cost-minimization holds for every (finite) set of observations generated by this factor demand function, then the factor substitution matrix must be negative semi-definite. You may use a related result in mathematics without proof, but you need to state that result clearly.

150 Werner

Question I.2 Spring 2013 majors

Suppose that there are S states of nature with known probabilities $\pi_s > 0$ for each s An agent's utility function U on state-contingent consumption plans is $U(c) = \min_s c_s$

(a) Derive this agent's risk compensation $\rho(w, z)$ for any risky claim $z \in \mathbb{R}^S$ with $E(z) = 0$ at any risk-free wealth $w \in \mathbb{R}_+$

(b) Show that this agent's risk compensation is greater than the risk compensation of every agent (risk averse, or not) whose preferences have expected-utility representation with strictly increasing utility.

(c) Suppose that there are two assets: a risk-free asset with state-independent (gross, or per-dollar) return $\bar{r} > 0$ and a risky asset with return $\tilde{r} = (r_1, \ldots, r_S)$. Assuming that $\min_s r_s < \bar{r} < \max_s r_s$ show that the optimal investment in the risky asset for this agent is zero for any risk-free initial wealth w

151 Allen

Question II.1 Spring 2013 majors

Consider a pure exchange economy with ℓ goods and n traders, each having consumption set $\mathbb{R}^\ell_+$, initial endowment $e_i \in \mathbb{R}^\ell_+$, and preferences $\preccurlyeq_i$ which are assumed to be complete continuous preorders on $\bar{R}^\ell_+$.

(a) Define competitive equilibrium in this economy using this notation.

Define a budget based competitive equilibrium or BBCE as follows:

A BBCE is an n -tuple $((x_1^*, B_1^*), (x_2^*, B_2^*), \ldots, (x_n^*, B_n^*))$ with $x_i^* \in B_i^* \leqq \mathbb{R}^\ell_+$ for all $i = 1, 2, \ldots, n$ such that

(0) $x_i^* \in \mathbb{R}^\ell_+$ for all i (automatic from the notation above defining a BBCE)
(1) x_i^* maximizes $\ll_i$ on B_i^*
(2) $\sum_{i=1}^n x_i^* = \sum_{i=1}^n e_i$(so no free disposal)

To avoid confusion below, a competitive equilibrium will now be called a price based competitive equilibrium, abbreviated PBCE.

(b) Prove that every such economy has a BBCE. (Hint: This follows trivially from a simple observation.)

(c) Prove that every PBCE gives rise to a BBCE in a natural way so that we could say that any PBCE "is" a BBCE.

(d) Using (c), state a reasonably general set of sufficient conditions on preferences to guarantee that every such economy has at least one BBCE. Explain your answer briefly.

(e) Suppose that $((x_1^*, B_1^*), \ldots, (x_n^*, B_n^*))$ is a BBCE. Show that, under your assumptions in (d) above, it cannot be the case that any B_i^* is open in $\mathbb{R}^\ell_+$, at least for generic $e_i \in \mathbb{R}^\ell_+$

(f) Assume that $e_i \gg 0$ and that $\preccurlyeq_i$ is strictly monotone for all $i = 1, \ldots, n$. Prove (again this almost trivially follows from a bit of mathematical analysis and a simple observation) that every BBCE that "is" a PBCE gives rise to uncountably many additional distinct BBCEs.

(g) Under what conditions on the B_i^* and on $\preccurlyeq_i$ is it true that every BBCE is Pareto optimal? Your conditions should be more general than the BBCE being a PBCE. Sketch a proof of your claim.

(h) What common economic phenomena could be modeled using BBCE as the equilibrium concept? Discuss.

152 Allen

Question II.2 Spring 2013 majors

This question concerns the generic approach to the study of smooth (pure exchange) economies.

(a) Define generic.

(b) Define a smooth pure exchange economy. (Your answer should introduce some notation that will be used throughout this question.)

(c) Define the regular and critical points and the regular and critical values of a C^1 function F from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(d) State the Regular Value Theorem.

(e) State Sard's Theorem.

(f) State Debreu's result (Econometrica 1970) on the number of equilibria of smooth pure exchange economies.

(g) Discuss the economic significance of the result you stated in part (f).

(h) Discuss the advantages and disadvantages of analyzing economic equilibrium through the study of generic properties of smooth economies (versus pure exchange economies that aren't necessarily assumed to be smooth).

(i) How could these results be extended to private ownership economies with production? What would you add to the basic model of pure exchange economies and what assumptions about the production sector would be needed?

153 Rustichini

Question III.1 Spring 2013 majors

A two players finite action normal form game is z ero sum if the sum of the utilities of the two players is equal to 0 for any action profile. Let $u^1 = u$, so $u^2 = -u$. The minimax theorem states that in this case

$$\min_{\alpha^2 \in \Delta(A^2) \alpha^1 \in \Delta(A^{\pm})} u\left(\alpha^1, \alpha^2\right) = \max_{\alpha^1 \in \Delta(A^{\pm}) \alpha^2 \in \Delta(A^2)} u\left(\alpha^1, \alpha^2\right) \equiv v$$

(a) Prove the minimax theorem; you can use Nash equilibrium existence theorem.

(b) Prove that the function V from u to v is continuous.

(c) State and prove the properties of the function that you can identify. In particular, is V more than just continuous?

154 Rustichini

Question III.2 Spring 2013 majors

Which of the following is true, if any? Answer the question and for each of the two statements, give a proof or a counterexample

(a) A subgame perfect equilibrium is sequential.

(b) A sequential equilibrium is subgame perfect.

155 Rustichini

Question IV.1 Spring 2013 majors

In answering this question you may assume that action sets and public signals set are finite.

(a) Define repeated game with imperfect public monitoring. Prove that every such has an equilibrium in public behavioral strategies.

(b) A Markov strategy in a repeated game with imperfect public monitoring is a behavioral strategy that depends only on the value of the public signal in the past period. Prove that there is an equilibrium in Markov strategies.

(c) Give an example of a game and an equilibrium which is in public behavioral strategies which is not Markov.

156 Rustichini

Question IV.2 Fall 2013 majors

(a) Define the set of Nash equilibrium payoffs of an infinitely repeated game with discounting δ; call it NEP^{δ}

(b) Characterize the set as a fixed point of an operator on subsets of F^*

(c) State and prove the basic properties of such operator that you are able to formulate.

(d) Give an example of an infinitely repeated game with Nash equilibrium payoff that is not sub-game perfect.

157 Werner

Question I.1 Fall 2013 majors

Consider preference relation $\succeq$ on the consumption set $\mathbb{R}^L_+$. Suppose that $\succeq$ is reflexive and complete.

(a) State a definition of $\succeq$ having a utility representation. Is utility representation, if it exists, unique?

(b) State a theorem providing sufficient conditions on $\succeq$ to have a utility representation. Be as general as you can and clearly define any extra properties of $\succeq$ that you use.

(c) Prove your theorem assuming additionally that preference relation $\succeq$ is strictly increasing (i.e., strongly monotone).

158 Werner

Question I.2 Fall 2013 majors

(a) In what sense is it true that risk compensation identifies a von Neumann-Morgenstern (or Bernulli) utility function? State the result clearly.

(b) Prove the result from part (a). If your proof relies on the Theorem of Pratt, you may use it without proving the theorem, but you need to state it clearly.

(c) Apply your result from part (a) to risk compensations that do not depend on (initial) wealth. Provide a characterization of utility functions with wealth-independent risk compensation.

159 Allen

Question II.1 Fall 2013 majors

Consider a pure exchange economy with l goods and n traders, each having consumption set $\mathbb{R}^L_+$, initial endowment $e_i \in \mathbb{R}^L_+$, and preferences $\succeq$ which are assumed to be complete continuous preorders on $\mathbb{R}^L_+$.

(a) Define competitive equilibrium in *this* economy using *this* notation. Define a budget based competitive equilibrium or BBCE as follows:

A BBCE is an n-tuple $((x_1^*, B_1^*), (x_2^*, B_2^*), \ldots, (x_n^*, B_n^*)$ all $i = 1, 2, \ldots n$ such that

(0) $x_i^* \in \mathbb{R}^L_+$ for all i (automatic from the notation above defining a BBCE)
(1) x_i^* maximizes $\succeq$ on B_i^*
(2) $\sum_{i=1}^n x_i^* = \sum_{i=1}^n e_i^*$ (so no free disposal).

To avoid confusion below, a competitive equilibrium will now be called a price based competitive equilibrium, abbreviated PBCE.

(b) Prove that every such economy has a BBCE. (Hint: This follows trivially from a simple observation.)

(c) What does your proof in part (b) tell us about the potential usefulness for economics of this statement about the existence of BBCE or, indeed, the definition of BBCE? In particular, does your result guarantee that there are equilibria of economic interest?

(d) Prove that every PBCE gives rise to a BBCE in a natural way so that we could say that any PBCE "is" a BBCE.

(e) Is it true that every BBCE allocation is Pareto optimal? Explain why or why not.

(f) Is it true that every Pareto optimal allocation in the economy is a BBCE allocation for some BBCE? Explain why or why not. For your answer, you may make additional standard "textbook" assumptions that appear in the second welfare theorem, but ifyou do so, you must state the additional assumptions that you are making and briefly indicate why each additional assumption is needed.

(g) Does your answer to part (f) change if we are allowed to reallocate individual initial endowment vectors in the economy (of course keeping the total endowment vector the same)? Explain your reasoning.

160 Allen

Question II.2 Fall 2013 majors

This question concerns nonconvexities in pure exchange economies. Suppose we have an economy with l commodities and n traders, $i = 1, 2, \ldots, n$, each having initial endowment vector $e_i \mathbb{R}^L_{++}$ and preferences $\succeq_i$ which are assumed throughout to be continuous complete preorders on $\mathbb{R}^L_+$.

(a) Define weak convexity of $\succeq_i$.

(b) State and define an assumption that guarantees that a weakly convex preference $\succeq_i$ is necessarily convex. Prove your assertion.

(c) Using this notation, define competitive equilibrium.

(d) If preferences are not assumed to even be weakly convex, what goes wrong in the standard fixed point argument that is used to prove existence of competitiveequilibrium? Be very precise in stating which of the (sufficient) conditions on(which?) correspondence for the application of a (which one? state it!) fixed pointtheorem are violated.

(e) Suppose that $\succeq_i$ are convex for all i but each consumer has a feasible consumption set $X_i \subseteq \mathbb{R}^L_+$, where $e_i \in X_i$ but X_i is not necessarily convex, although it is closed. What goes wrong in the standard fixed point argument that is used to prove existence of competitive equilibrium? As in part (d), identify a correspondence and state which of the (sufficient) conditions for your fixed point theorem would be violated.

(f) In part (e), why did we assume at the beginning that the $succeq_i$ were defined on $\mathbb{R}^L_+$ rather than X_i?

(g) What happens if $\succeq_i$ are not necessarily convex and consumers have feasible consumption sets $X_i \subseteq \mathbb{R}^L_+$ (with $e_i \in X_i$) where the X_i are closed but not necessarily convex?

(h) What could happen if the X_i were not necessarily closed (but they are convex and preferences are all convex), but as before, $e_i \in X_i \subseteq \mathbb{R}^L_+$ for all i? You may assume that preferences are strictly convex if you think it helps, but this probably will not help you to answer this question.

(i) Give a few examples of economic phenomena that could be described by the situations analyzed here.

161 Rustichini

Question III.1 Fall 2013 majors

(a) Define (i) the Best Response Correspondence for a Normal Form Game, (ii) a perturbation η of the mixed strategy set in a normal form game, as introduced in thedefinition of perfect equilibria.

(b) For a perturbation η, let u_η be the utility function induced by the perturbation. Let $BR^i_{T^i}(s, u^i)$ be the best response to s of player i with utility ui and strategy set $T^i \subseteq S^i$. What is the connection between $BR^i_{S^i_\eta}(s, u^i)$ and $BR^i_{S^i}(s, u^i_\eta)$

(c) Prove the existence of perfect equilibria of normal form games

162 Rustichini

Question III.2 Fall 2013 majors

(a) Find all the transformations of the utility functions of a player that leave the Best Response correspondence of that player unchanged.

(b) Consider all $2 \times 2 \times 2$ normal form games, that is games with two players, each player with two actions. Illustrate the result in the first part (a) above in these games.

(c) Introduce a distance among all $2 \times 2 \times 2$ normal form games, and consider all games with the property that all other games at a distance less than some $\epsilon > 0$ have the same number and type (mixed or pure) of equilibria. You can now partition these games into subsets, each with the same number and type of equilibria. Characterize these subsets.

163 Rustichini

Question IV.1 Fall 2013 majors

In answering this question you may assume that action sets and public signals set are finite.
(a) Define repeated game with imperfect public monitoring and public behavioral strategies. Prove that every repeated game with imperfect public monitoring has an equilibrium in public behavioral strategies.

(b) A Markov strategy in a repeated game with imperfect public monitoring is a behavioral strategy that depends only on the value of the public signal in the past period. Prove that there is an equilibrium in Markov strategies.

(c) Give an example of a game and an equilibrium is in public behavioral strategies which is not Markov.

164 Rustichini

Question IV.2 Fall 2013 majors

(a) Consider a pure strategy Nash equilibrium $\hat{s}$ of an extensive form game. Prove that the strategy profile induces an equilibrium in every subgame that is reached by $\hat{s}$.

(b) Define an extension of the concept of backward induction for a game of perfect information with a countable set of nodes, not necessarily finite.

(c) Does this procedure define an equilibrium in pure strategies? Prove your claim in detail.

165 Werner

Question I.1 Fall 2013 minors

Consider a production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ with n inputs and one output. Assume that $f(0) = 0$.
(a) State a definition of f having (strictly) increasing returns to scale.

(b) Prove that if f exhibits increasing returns to scale, then, for any strictly positive input prices w_i (where $i = 1, \ldots, n$) and strictly positive output price p, either the firm's output at the profit-maximizing production plan is zero or otherwise the profit-maximizing production plan is not well defined (i.e. it does not exist).

(c) Consider the following example of production function with two inputs:

$$f(x_1, x_2) = [\min x_1, x_2]^2$$

Does this f exhibit increasing returns to scale?

(d) Does the cost-minimization problem for production function f of (c) have a solution for arbitrary prices $w_1 > 0, w2 > 0$ and output level $y > 0$? Justify your answer

166 Rustichini

Question I.2 Fall 2013 minors

Describe the Ellsberg paradox. Show that the pattern of preferences in the Ellsberg paradox is inconsistent with any expected utility function. Give an example a preference relation (or a utility function) defined on the set of bets (or gambles) of arbitrary amounts of money on a ball of any color that is consistent with the pattern in the Ellsberg paradox.
Justify your answer.

167 Allen

Question II.1 Fall 2013 minors

In pure exchange economies in which consumers may have nonconvex preferences, the potential presence of such nonconvexities can cause one of the welfare theorems not to hold.

(a) State both welfare theorems clearly. Be sure to begin by defining notation for a pure exchange economy, defining its competitive equilibria and its Pareto optimal allocations, and specifying what assumptions are needed for your definitions to make sense.

(b) Prove the welfare theorem that does not require convexity.

(c) For the welfare theorem that does require convexity, indicate where convexity of preferences is needed in the proof. Explain.

168 Allen

Question II.2 Fall 2013 minors

Consider a pure exchange economy under uncertainty with asymmetric information where expected utility maximizers have state-dependent utility functions and prices may depend on states of the world. In particular, consider an example with two commodities (x and y), two traders, and three equally-probable states of the world denoted L, M, and H. [Think of low, medium and high temperature where x is ice cream and y is hot coffee.] Each agent (denoted by subscripts 1 and 2) has an initial endowment vector which is independent of the state, in particular, $e_1(L) = e_1(M) = e_1(H) = (1, 1)$ and $e_2(L) = e_2(M) = e_2(H) = (1, 1)$. Suppose that agent 1 knows whether the state is L or is 'M or H', and has state-dependent cardinal utilities given by the following:

$$u_1(x, y; L) = 1/3 \log x + 2/3 \log y$$

$$u_2(x, y; M) = u_2(x, y; H) = 2/3 \log x + 1/3 \log y.$$

Agent 2 knows whether the state belongs to {L, M} or is H. Utilities for agent 2 are as follows:

$$u_2(x, y; L) = u_2(x, y; M) = 1/3 \log x + 2/3 \log y$$

$$u_2(x, y; H) = 2/3 \log x + 1/3 \log y$$

. In each state of the world, normalize prices to sum to 1.

(a) Calculate state-dependent demands for each agent.

(b) Calculate the competitive equilibrium under uncertainty.

(c) Is it Pareto optimal? Define and explain what Pareto optimality means in this context.

(d) Are your equilibrium prices state dependent? Can either agent 1 or agent 2 or both learn from prices? Can they learn payoff relevant information that they did not have initially?

(e) Is the equilibrium you found in part (b) a rational expectation equilibrium? Explain.

169 Rustichini

Question III.1 Fall 2013 minors

This question has two steps.
Step 1 (a) Find all the sets that survive iterated elimination of weakly dominated strategies in the following game:

	l	m	r	e
T	0,0	2,2	0,0	-2,4
M	6,2	4,8	-8,2	2,6
B	-2,-4	4,-6	-2,2	-2,8

Be sure that you consider all possible sequences of elimination.

(b) Find all the sets that survive iterated elimination of strictly dominated strategies in the same game

Step 2

Below you will find several normal form games; these are exactly those for which you found the Nash equilibria in the past. For each of these games find:
(a) The set of Nash equilibria and Nash equilibrium payoffs for the stage game

(b) The set $co(F^0)$

(c) The vector of minimax values $(v^i)_{i\in I}$

(d) The set of feasible and individually rational payoffs

The games:

(a)

	l	r
T	-2,2	-3,3
B	4,-4	-1,1

(b)

	l	r
T	6,1	1,1
B	-1,1	1,6

(c)

	l	r
T	4,3	1,2
B	2,1	1,7

170 Rustichini

Question III.2 Fall 2013 minors

(a) Find all the transformations of the utility functions of a player that leave the Best Response correspondence of that player unchanged.

(b) Consider all $2 \times 2 \times 2$ normal form games, that is games with two players, each player with two actions. Illustrate the result in the first part (a) above in these games.

(c) Introduce a distance among all $2 \times 2 \times 2$ normal form games, and consider all games with the property that all other games at a distance less than some $\epsilon > 0$ have the same number and type (mixed or pure) of equilibria. You can now partition these games into subsets, each with the same number and type of equilibria. Characterize these subsets.

171 Rustichini

Question IV.1 Fall 2013 minors

(a) Write the extensive form for the repeated game which is repeated for two periods, with $\delta = 1$, and has stage game

	l	r
T	4,1	5,6
B	2,0	-1,4

(b) Find the subgame perfect equilibria of the repeated game. Prove your answer in detail

172 Rustichini

Question IV.2 Fall 2013 minors

Consider the infinitely repeated game with stage game given by the PD game, and consider the following tentative agreement:

Play (C, C) in every period as long as both players have played (C, C) in the past. After any deviation, play (D, D) for two periods, independently of the past history, then go back to play (C, C), as long as both players have played (C, C) in all periods following the first deviation.

For which, if any δ is this agreement a subgame perfect equilibrium?

173 Werner

Question I.1 Spring 2014 majors

Consider a firm that has two technologies described by production functions $f_1 : \mathbb{R}^n_+ \to \mathbb{R}_+$ and $f_2 : \mathbb{R}^n_+ \to \mathbb{R}_+$ that turn n inputs in two different output goods. The firm has fixed amounts of inputs $\omega \in \mathbb{R}^n_{++}$ that can be used in production. The firm maximizes profit by solving the following optimization problem:

$$\max_x p_1 f_1(x) + p_2 f_2(\omega - x)$$

$$\text{subject to } 0 \leq x \leq \omega;$$

where $p_1 > 0$ and $p_2 > 0$ are output prices. Let $x^*(p_1; p_2)$ be a solution (assumed unique). (i) State a definition of production function $f_i(x)$ being supermodular in x: Give an example of strictly increasing function f_i that is not supermodular. Justify your answer.

(ii) Show that, if production functions f_1 and f_2 are strictly increasing and supermodular in x, then $x^*(p_1; p_2)$ is non-decreasing in p_1. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

174 Werner

Question I.2 Spring 2014 majors

There are two assets: a risk-free asset with (gross) return $\bar{r}$ and a risky asset with return r that takes value r_s in each state of nature $s \in S$ where S is a finite set. Consider an agent who allocates her risk-free initial wealth $w > 0$ between those two assets so as to maximize the expected utility of investment payoff. The payoff of investing a dollars in the risky asset and $w - a$ in the risk-free asset is $((w - a)\bar{r} + ar)$: Investment a can be positive or negative. The agent is strictly risk averse. Further, her von Neumann-Morgenstern (or Bernoulli) utility function v is strictly increasing and differentiable.

(i) Show that the optimal investment in the risky asset is strictly positive, zero or strictly negative if and only if the risk premium on the risky asset (i.e., $E(r) - \bar{r}$) is, respectively, strictly positive, zero or strictly negative. $E(r)$ denotes the expected return on the risky asset. You may use any well-known characterization of risk-averse utility functions without proof.

(ii) Suppose that the agent is facing some background risk w_1 at the time of payoffso that her total wealth after payoff is $w_1 + (w - a)\bar{r} + ar$; where w_1 is state dependent. Does the result from part (i) extend to this case of background risk? Justify your answer.

175 Allen

Question II.1 Spring 2014 majors

It was asserted in class (but not proved) that in a textbook pure exchange economy, competitive equilibria are unique whenever initial endowments are Pareto optimal.

(a) Introduce notation and define a pure exchange economy. Make (state them precisely) minimal assumptions needed, then defi
ne competitive equilibria and Pareto optimality. State and prove the first welfare theorem.

(b) Give an example (a clear Edgeworth box is sufficient - do not take time to specify explicit utility functions) to show that, with nonconvex preferences, it can be the case that there is an additional competitive equilibrium allocation even when the initial endowment is Pareto optimal.

(c) Now assume that preferences are complete continuous preorders that are monotone and convex. Assume also that initial endowments are Pareto optimal.

(i) Are competitive equilibrium allocations necessarily unique? Explain briefly if your answer is yes; give an Edgeworth box counter-example if your answer is no.

(ii) Are normalized competitive equilibrium prices necessarily unique? Explain briefly if your answer is yes; give an Edgeworth box counter-example if your answer is no.

(d) What additional assumptions are needed, beyond those in part (c), to guarantee uniqueness of competitive equilibrium allocations, whenever initial endowments are Pareto optimal? Explain briefly.

(e) What additional assumptions, beyond those in part (c), are needed to guarantee that normalized competitive equilibrium prices are unique? Explain briefly.

176 Allen

Question II.2 Spring 2014 majors

In general equilibrium models of pure exchange economies, in principle we can discuss the extent to which one specified commodity is the substitutable for another (different) specified commodity. As in intermediate microeconomics, we can examine the marginal rate of substitution. (a) Deffine the marginal rate of substitution for a consumer with preferences $\succeq$ deffined on $\mathbb{R}^2_{++}$. Be sure to specify precisely what assumptions are needed to make your deffinition make sense.

(b) Prove that your deffinition in part (a) depends only on the preferences of the agent and not the utility function chosen to represent those preferences.

(c) While we can (under some assumptions) generalize the deffinition in (a) to apply to economies with infinitely many commodities, we often instead use a metric on the commodity space to indicate whether two commodities are close substitutes for each other. This has the "advantage" of not depending on the individual agent (at least under certain assumptions) or the point in the commodity space that we are considering. Specify those assumptions and briefly discuss the statement.

(d) Suppose that we want to be able to make sense of the property that commodity A is a closer substitute to commodity B than commodity A is to commodity C. For an individual consumer, carefully deffine this concept using marginal rates of substitution and again using a metric on the commodity space. What assumptions on preferences are needed for your deffinitions to make sense, and what assumptions are needed to capture what we mean by substitutability between commodities?

(e) Alternatively, if the commodity space (finite or infinite) is a subset of a topological vector space (so that convex combinations are defined), we could say that A is a closer substitiute for B than A is a substitute for C, if it is the case that B is a convex combination of A and C (or intuitively, B lies between A and C). Can you state conditions on preferences such that this alternative definition would capture what we mean by substitutability?

177 Rustichini

Question III.1 Spring 2014 majors

Consider a normal form game with a finite number of players, where every player has a finite actions set.

(a) Prove that the set of Nash Equilibria is closed.

(b) Two games are *strategically equivalent* if the best response correspondence of each player is the same in both games. Suppose the game has two players, and each player has two actions, {T;B} and {L;R} respectively. Prove that any such game is strategically equivalent to a game where the off diagonal payoffs (that is, the payoffs for the action profiles (T;R) and (B; L)) are zero for both players.

178 Rustichini

Question III.2 Spring 2014 majors

Consider a Normal form game with a finite number of players, where every player has a finite actions set.
(a) Define mixed strategies and correlated strategies;

(b) Prove that the set of correlated strategies induced by mixed strategies is closed;

(c) Is this set convex? Prove or disprove your statement.

179 Rustichini

Question IV.1 Spring 2014 majors

Consider the Game:

	l	r
T	5,5	2,7
B	7,2	-4,4

and consider the repeated game with this stage game, with discounting $\delta \in (0; 1)$

(a) Find a Nash equilibrium that is not a Subgame perfect equilibrium. Prove in detail both that it is a Nash and that it is not a Subgame Perfect equilibrium

(b) Prove that the restriction of the Nash equilibrium strategy profile to the equilibrium history is a Nash equilibrium.

(c) Do the same as parts (a) and (b) above for the repeated game with this stage game:

	l	r
T	20,2	20,0
B	30,30	0,60

180 Rahman

Question IV.2 Spring 2014 majors

Moral Hazard. Consider a two-period principal-agent problem. In each period, there are m possible output levels for the principal,$\pi_i \in R$, and n effort levels for the agent, $e_j \in R$. Let $Pr(\pi|e)$ be the probability of π given e. The principal's per-period payoff is expected output minus payments to the agent. The agent's is

$$U(z;a) = v(z) - c(a);$$

where $v' > 0$, $v'' < 0$, $c' > 0, c'' > 0$, $v(z)$ corresponds to the agent's utility from payments by the principal and $c(a)$ corresponds to effort costs. Time elapses as follows. First, the principal makes a take-it-or-leave-it offer to the agent, who has an outside option worth $\bar{U}$.

Once everyone agrees to the contract, there are no more opportunities to quit throughout the two-period relationship. In period 1, the agent exerts effort e_1, output π_1 realizes, and the agent is paid $w_1(\pi_1)$. Everyone observes π_1 at the end of period 1. In period 2, the agent exerts effort e_2, output π_2 realizes, and the agent is paid $w_2(\pi_1;\pi_2)$. Everyone's overall payoff is the sum of per-period payoffs. (a) Write down formally the principal's problem of (i) minimizing the cost of implementing a given effort profile $e = (e_1; \hat{e_2})$, where $e_1 \in R$ is period-1 effort and $\bar{e_2} \in R^m$ is the effort plan in period 2 contingent on output in period 1 subject to a lifetime participation constraint, and (ii) maximizing expected profit.
(b) Show that, in general, w_2 will depend not only onπ_2, but also on π_1. (c) Derive and interpret the inverse Euler equation from (a-i):

$$\frac{1}{v'(w_1(\pi_1))} = \sum_{\pi_2} \frac{1}{v'(w_2(\pi_1;\pi_2))} Pr(\pi_2|e_2(\pi_1)) \quad \forall \pi_1$$

181 Rahman

Question IV.3 Spring 2014 majors

Screening. A monopolist faces a single consumer with utility function $u = \theta q \frac{1}{2} q^2 - T$, where$\theta$ is private information of the consumer, q is the level of consumption and T is the amount of money that the consumer pays the monopolist. The monopolist's cost of producing q equals $\frac{1}{2}cq^2$ for some constant $c > 0$. The consumer's reservation utility equals 0. The (Pareto) CDF ofθ is $F(\theta) = 1 - \theta^{\alpha}$ for all $\theta \in [1;\infty)$, where $\alpha > 1$.

(a) Derive the monopolist's optimal bundle (q; T) assuming that it knows θ.

(b) Write down the monopolist's nonlinear pricing problem.

(c) Derive the optimal nonlinear pricing schedule and compare it with (a).

(d) What can you say about distortions at the top? (Here top means $\theta \to \infty$.)

182 Werner

Question I.1 Fall 2014 majors

There are two conditions often used to deffine continuity of a preference relation$\succeq$ on consumption set $X = \mathbb{R}^L_+$:

(A) for every sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_n x^n = x$; $\lim_n y^n = y$; and $x^n \succeq y^n$; it holds $x \succeq y$:

(B) For every $x \in X$; the preferred-to-x set $\{y \in X : y \succeq x\}$; and the lower contour set $\{y \in X : x \succeq y\}$ are closed.

(i) Assuming that$\succeq$ is transitive and complete, prove that conditions (A) and (B) are equivalent.

(ii) Give an example of a transitive and complete preference relation on $\mathbb{R}^2_+$ for which the preferred-to-x sets are closed for all x, but the lower contour sets are not closed for some x.

183 Werner

Question I.2 Fall 2014 majors

Let $\hat{y}$ and $\hat{z}$ be arbitrary random variables on some finite state space. Let $E(\hat{y}) = E(\hat{z})$ where E denotes expected value under the given probabilities on states.

(i) State a definition of $\hat{y}$ being more risky than $\hat{z}$ in terms of cumulative distribution functions of $\hat{y}$ and $\hat{z}$. Prove that if $\hat{y}$ is more risky than $\hat{z}$, then

$$E[v(\hat{z})] \geq E[v(\hat{y})]$$

for every non-decreasing, continuous and concave utility function $v : \mathbb{R} \to \mathbb{R}$.

(ii) Show that, if $\hat{y}$ is more risky than $\hat{z}$, then $var(y) \geq var(z)$, where var denotes variance. (iii) Show that, if $\hat{y}$ is more risky than $\hat{z}$, then $w + \hat{y}$ is more risky than $w + \hat{z}$; for every deterministic w: Here, you may use without proof the fact that the necessary condition for more risky from part (i) is also sufficient.

184 Allen

Question II.1 Fall 2014 majors

Suppose that a pure exchange economy has n traders and let $N = \{1, 2, \dots n\}$ be the set of traders. Each consumer $i \in N$ has consumption set $\mathbb{R}^l_+$, initial endowment $e_i \in \mathbb{R}^l_{++}$ and a utility function $u_i : \mathbb{R}^l_+ \to \mathbb{R}$.

(a) State a set of minimal assumptions needed to define competitive equilibrium and Pareto optimality. Briefly explain why each assumption is needed or, if no assumption is needed, say so and briefly explain.

(b) Define competitive equilibrium in this economy using this notation.

(c) Define weak Pareto optimality and strong Pareto optimality in this economy using this notation.

(d) Again for this economy using this notation, state the first welfare theorem.

(e) Prove the first welfare theorem (as you have stated it).

(f) Now suppose that each ui is an expected utility for an economy with contingent commodity contracts. To simplify suppose that there are two states of the world, called H and T and occurring with known probabilities $\pi(H)$ and $\pi(T)$ where $0 \leq \pi(H) \leq 1$, $0 \leq \pi(T) \leq 1$, and $\pi(H) + \pi(T) = 1$. Assume that l is an even (why?) strictly positive integer and interpret the first $\frac{l}{2}$ commodities as those delivered in state H with the remaining goods delivered in state T. Then write state-dependent utilities as

$$v_{i,H} : \mathbb{R}^{l/2}_+ \to \mathbb{R} \qquad v_{i,T} : \mathbb{R}^{l/2}_+ \to \mathbb{R}$$

so that we have (for $x_i \in \mathbb{R}^l_+$)

$$u_i(x_i) = \pi(H)v_{i,H}(x_{i,1}, x_{i,2}, \dots x_{i,l/2}) + \pi(T)v_{i,T}(x_{i,l/2+1}, \dots, , x_{i,l})$$

Under what assumptions (if any) on $\pi(H)$, $\pi(T)$, and the $v_{i,H}$ and $v_{i,T}$ functions does the first welfare theorem apply? Explain your answer.

185 Allen

Question II.2 Fall 2014 majors

In microeconomic theory, we tend to make some assumptions purely for convenience (to simplify spefications of the model or to shorten proofs) and others because we know them to be necessary and sufficient conditions for the desired result. Most fall into the middle ground of being somewhat helpful for the model, proof, etc. but we know they're not really needed, yet they seem somewhat inconsequential because we expect the assumption either to be satisfied in the situation of interest or to be viewed as something that isn't closely related to our main focus. This question first asks you to explore some cases where the presence of a few 'weird' agents do not destroy our standard results on existence of competitive equilibrium and welfare theorems. Assume that the economy has only two commodities with consumption sets being $\mathbb{R}^2_+$. Initial endowments will be denoted by $e_i \in \mathbb{R}^2_+$. It's easiest to answer these questions with clearly drawn pictures of endowments and indifference curves.

(a) Give an example of a preference relation $\succeq_i$ which is convex but not strictly convex, and an initial endowment vector $e_i \in \mathbb{R}^2_+$ such that the consumer's demand is single valued for all nonnegative price vectors in $\mathbb{R}^2_+$. In this case, demand will be a continuous (and in fact smooth) function so that the use of the Easy Existence Theorem can still be valid. In fact, identify the class of preference relations for which this 'trick' works for the initial endowment vector you specifi
ed, and explain briefly.

(b) Now assume that $e_i \in \mathbb{R}^2_{++}$. Find an example with monotone and convex but not strictly convex preferences so that demand is a single-valued continuous function for all strictly positive price vectors $p \in \mathbb{R}^2_{++}$.

(c) Again take $e_i \in \mathbb{R}^2_{++}$ and find a preference relation which is lns (but not monotone) and convex (but not strictly convex) such that again demand is a single valued continuous function for all nonnegative price vectors $p \in \mathbb{R}^2_+$. [Hint: You can modify an example that works for part (b).]

(d) For $e_i \in \mathbb{R}^2_{++}$ and preferences that are weakly convex (but not convex) and not even lns fi
nd a preference relation such that demand is set valued for all nonnegative prices but there is a continuous single-valued selection from the agent's demand correspondence.
(e) Take a standard pure exchange economy with n traders such that $e_i \in \mathbb{R}^2_{++}$ for all i and each trader's preference relation $\succeq_i$ is a continuous complete preorder which is strictly monotone and strictly convex. Now add m traders to this economy with $e_i \in \mathbb{R}^2_{++}$ having the property that the initial endowment vector belongs to the individual demand correspondence for all strictly positive prices. Show (explain your reasoning clearly) that the Easy Existence Theorem can be used to prove that the augmented economy has at least one competitive equilibrium. What can you say about the comparison between competitive equilibria in the original and the augmented economy? Explain briefly.

(f) State the Sonnenschein conjecture on the characterization of aggregate excess demand in pure exchange economies with preferences satisfying strict convexity and some other assumptions (state them). How can you modify the theorem if preferences are convex but not strictly convex? Discuss whether the result for convex but not strictly convex preferences should be considered to be weaker or stronger.

186 Rustichini

Question III.1 Fall 2014 majors

A two players finite game is called strictly competitive if for every pair a and b of action profiles

$$u_1(a) > u_1(b) \quad \text{if and only if} \quad u_2(a) < u_2(b)$$

(a) Give an example of a strictly competitive game that is not constant sum game.

(b) Lat for each player i $maxmin_i \equiv max_{a^i} min_{a^{-i}} u^i(a^i; a^{-i})$, and $minmax_i \equiv min_{a^{-i}} max_{a^i} u^i(a^i; a^{-i})$. Prove the following:
For a strictly competitive game

(a) If for i = 1; 2 we have $maxmin_i = minmax_i$, then G has a Nash equilibrium.

(b) If G has a Nash equilibrium, then for for $i = 1, 2$ we have $maxmini = minmaxi$.

187 Rustichini

Question III.2 Fall 2014 majors

(a) State precisely Kuhn's theorem. Please provide de
finitions of the strategy sets you are using. If you use a notion of equivalence, state its definition.

(b) Provide an example to show why the restriction on the games for which the conclusion of the theorem holds cannot be dispensed with.
(c) Consider an extensive form game with two players who alternate in the choice of moves (player 1 moves in even periods, players 2 moves in odd periods); each player has two actions in every period.

(a) State Kuhn's theorem in this case.

(b) Prove the theorem you state.

188 Rahman

Question IV.1 Fall 2014 majors

There are n individuals considering whether or not to place a sculpture (already sculpted, so it costs nothing) in the park. Each individual i has utility function

$$u_i(k; t_i) = v_i k + t_i$$

for some number $v_i \in R$, where $k \in \{0, 1\}$ denotes whether (1) or not (0) the statue is placed in the park. The number v_i reflects the taste of individual i for the sculpture which in principle can range anywhere in R, and the quantity $t_i \in R$ is a monetary transfer.

(a) Describe a Vickrey-Clarke-Groves (VCG) mechanism for this economy formally and intuitively.

(b) Show that every VCG mechanism fails to satisfy budget balance, that is, there exists a profile $v = (v_1, \ldots v_n)$ at which the monetary transfers $t = (t_1, \ldots, t_n)$ satisfy

$$\sum_{i=1}^{n} t_i(v) \neq 0$$

(c) Let $g(v) = \max\{\sum_i v_i, 0\}$ be the maximum possible welfare from optimally placing the sculpture or not when everyone's utility is given by v, and $k(v) = 1_{\{} \sum_i v_i > 0\}(v)$ the optimal decision, that is, the indicator function of whether or not the sum of utilities is positive. Furthermore, let $g_i(v, v_i') = k(v_i', v_{-i}) \sum_i v_i$ be the welfare when individual i misreports his utility to be v_i' instead of v_i and the optimal choice is made for reported-rather than actual—utilities.

(a) Find a VCG mechanism $(\bar{k}; \bar{t})$ such that

$$u_i(\bar{k}(v_i', v_{-i}), \bar{t}_i(v_i', v_{-i})) = g_i(v, v_i')$$

for all i, v_i, v_i' and v_{-i}

(b) Show that there is no mechanism $(\hat{k}, \hat{t})$ such that

$$u_i(\hat{k}(v_i', v_{-i}), \hat{t}_i(v_i', v_{-i})) = g_i(v, v_i')/n$$

for all i, v_i, v_i' and v_{-i}

(c) Is there another way of splitting g (other than equally as considered in part (b) above) to recover budget balance?

189 Rahman

Question IV.2 Fall 2014 majors

Consider the Prisoners' Dilemma with imperfect public monitoring.

	C	D
C	1,1	-1,2
D	2,-1	0,0

Flow Payoff

	C	D
C	p_2	p_1
D	p_1	p_0

PR(g)

Every period, players observe a binary signal $\omega \in \Omega = \{g, b\}$ that is IID conditional on players' actions. The bi-matrix on the left describes flow payoffs, and the matrix on the right describes the conditional probability of g, where $0 < p_0 < p_1 < p_2 < 1$. Players share a common discount factor $\delta < 1$. Assume that players have access to a public randomization device.

(a) Find the maximal strongly symmetric pure-strategy equilibrium payoff as $\delta \to 1$.

(b) Under what conditions on the parameters of the problem is this payoff feasible?

(c) Let $\delta = e^{-r\Delta t}$, where $r > 0$ is a discount rate and $\Delta t > 0$ is the length of time between interactions. Given x such that $x_0 < x_1 < x_2$, let $p_k = \frac{1}{2}[1 + x_k\sqrt{\Delta t}]$ for each k (therefore, the signals define a random walk with drift x that tends to Brownian motion as $\Delta t \to 0$), and assume that Δt is small enough that $0 < p_k < 1$ for all k. Redo parts (a) and (b) above

(i) as $r \to 0$ with Δt fixed,

(ii) as $\Delta t \to 0$ with r fixed. Discuss.

190 Werner

Question I.1 Spring 2015 majors

Consider demand function $d : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_{++}$ of prices and income satisfying budget equation $pd(p, w) = w$ for every p and w.

a Show that if d is a Walrasian demand function of a consumer with locally non-satiated utility function, then the Generalized Weak Axiom of Revealed Preference (GWARP) holds for every T-tuple of price-quantity pairs $\{p^t, x^t\}_{t=1}^T$ where $x^t = d(p^t, w^t)$, $p^t \in \mathbb{R}^L_{++}$ and $w^t \in \mathbb{R}_+$ for every $t = 1, \ldots T$. State GWARP.

b Consider the following example of demand function for $L = 2$:

$$\hat{d}(p, w) = \begin{cases} (\frac{w}{p_1}, 0) & \text{if} p_1 \geq p_2 \\ (0, \frac{w}{p_2}) & \text{if} p_1 < p_2 \end{cases}$$

Show that GWARP does not hold for $\hat{d}$.

c Suppose that GWARP holds for demand function d for L goods, but do not assume that d is a Walrasian demand function. Show that d satisfies the following law of compensated demand:

$$[d(p', w') - d(p, w)][p' - p] \leq 0$$

for every p, p', w, w' such that $w' = p'd(p, w)$.

191 Werner

Question I.2 Spring 2015 majors

Consider two real-valued random variables $\hat{y}$ and $\hat{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\hat{z})$ and $E(\hat{y})$ their expected values.

a State a definition of $\hat{z}$ first-order stochastically dominating (FSD) $\hat{y}$

b Show that, if $\hat{z}$ FSD $\hat{y}$ and $E(\hat{z}) = E(\hat{y})$; then $\hat{y}$ and $\hat{z}$ have the same distribution, i.e., $F_y(t) = F_z(t)$ for every $t \in \mathbb{R}$. If you find it convenient, you may assume in your proof that random variables $\hat{y}$ and $\hat{z}$ have densities, or alternatively that $\hat{y}$ and $\hat{z}$ are discrete random variables (i.e., take finitely many values).

c State a definition of $\hat{z}$ second-order stochastically dominating (SSD) $\hat{y}$: Show that if $\hat{z}$ SSD $\hat{y}$; then $E(\hat{z}) \geq E(\hat{y})$.

d State a definition of $\hat{y}$ being more risky than $\hat{z}$: Give a brief justification for why it is a sensible definition of more risky.

e Show that if $\hat{y}$ is more risky than $\hat{z}$; then $var(\hat{y}) \geq var(\hat{z})$ where $var(\cdot)$ denotes the variance. You may use a known characterization of more risky without proving it, but you should state it clearly.

192 Allen

Question II.1 Spring 2015 majors

Consider a pure exchange economy with l commodities and n traders, each having initial endowments $e_i \in \mathbb{R}^L_{++}$ and preferences $\succeq_i$ which areassumed to be complete continuous preorders on $\mathbb{R}^L_+$

(a) Define competitive equilibrium.

(b) Define the set of weakly Pareto optimal allocations.

(c) Define the set of strongly Pareto optimal allocations.

(d) State the First Welfare Theorem.

(e) Prove the theorem you stated in part (d).

(f) Suppose now that the economy has only indivisible commodities (available in integer amounts). How does this change the economic environment and assumptions in this question, including any assumptions you imposed in (d)?

(g) Is the First Welfare Theorem true with indivisible commodities? Explain why carefully (or give a counterexample). Also explain either how your proof in (e) must be modified or where it fails due to the indivisibilities.

(h) Would your answers above change if there were one perfectly divisible good? Explain briefly.

193 Allen

Question II.2 Spring 2015 majors

Write an essay on general equilibrium pure exchange economies with infinitely many commodities, focusing on (i) the set-up of the model, (ii) economic interpretations of various possible modeling choices, (iii) existence results, including assumptions, and (iv) welfare theorems, including assumptions. Be sure to discuss the economic implications of your assumptions in parts (iii) and (iv).

194 Rustichini

Question III.1 Spring 2015 majors

(a) Prove that a mixed strategy profile $\hat{s}$ in a finite normal form game is a Nash equilibrium if and only if for every player $i \in I$, where I is the set of players, $\hat{s}^i$ gives zero probability to an action b if b gives a strictly smaller payoff than some action a against $\hat{s}^{-i}$. More precisely:
$\hat{s}$ is a Nash equilibrium if and only if:

$$\forall i \in I \forall a, b \in A^i, \quad u_i(a, s^{-i}) > u_i(b, s^{-i}) \quad \Rightarrow s^i(b) = 0 \tag{194.1}$$

(b) Then define a a vector of perturbations and a perfect equilibrium of the normal form game;

(c) Write the condition that corresponds naturally to (1) when each player is choosing a strategy in the set constrained by a vector of perturbations;

(d) State a condition for a mixed strategy profile $\hat{s}$ which is necessary and sufficient for $\hat{s}$ to be a perfect equilibrium which corresponds to condition 1, and uses your answer to (c) above.

(e) Prove that the condition in your answer to (d) above is indeed necessary and sufficient.

195 Rustichini

Question III.2 Spring 2015 majors

(a) Find all the Nash equilibria of the following game:

	a	b	c
A	7,2	0,1	0,0
B	1,0	1,8	0,0
C	0,0	0,0	2,2

(b) Define canonical correlated equilibria; use the definition where the mediator communicates to each player only the player's recommended action, and only that.

(c) Find all the Correlated equilibria of the game;

(d) Is the set of Correlated equilibrium payoffs the convex combination of the set on Nash equilibrium payoffs? Prove your answer.

(e) Does your answer to part (d) change if the mediator communicates the entire action profile to each player?

196 Rahman

Question IV.1 Spring 2015 majors

Below are three contracting problems, where certain outcomes are to be implemented with ex post budget-balanced payment schemes, that is, such that the sum of payments across individuals always equals output. Call such payment schemes *partnerships*. Assume that agents have quasilinear utility with respect to money, as usual.

1. There are 3 agents. Each agent i can take two actions: $A_i = \{0, 1\}$. Let $A = \Pi_i A_i$ be the space of action profiles with generic elements a and b. Assume that the efficient action profile is $a^* = (1, 1, 1)$ but each agent i incurs a cost of $c_i > 0$ from choosing $a_i = 1$ instead of $a_i = 0$. Actions lead to output deterministically according to the function $y : A \to \mathbb{R}$. Let $y_1 = y(0, 1, 1)$, $y_2 = y(1, 0, 1)$ and $y_3 = y(1, 1, 0)$.

   (a) Assume that $y(a) \neq y(b)$ for all $a \neq b$; thus $y_i \neq y_j$ for all $i \neq j$. Find an efficient partnership: output-contingent payments that always add up to output and render incentive compatible the action profile $a^* = (1; 1; 1)$, that is, nobody wants to unilaterally deviate from a^* if everyone else is abiding by a^*.

   (b) Now assume instead that $y_2 = y_3 = \hat{y}$ but $y_1 \neq \hat{y}$. Is it possible to find an efficient partnership in this case?

   (c) What if $y_1 = y_2 = y_3$?

2. There are n agents, where $n \in \mathbb{N}$. Let $A_i = \mathbb{R}_+$ and $y(a) = \min\{a_1/\theta_1, \ldots, a_n/\theta_n\}$, where$\theta_i > 0$ is given. Each agent i has a disutility of effort function $c_i(a_i)$ that is strictly increasing and strictly convex, differentiable and satisfying $c_i'(0) = 0$.

   (a) Derive the first-order conditions that efficient output y^* must satisfy, as well as the efficient action of each agent a_i^* as a function of y^*

   (b) Show that the linear sharing rule given by $S_i(y) = \theta_i c_i'(\theta_i y^*)y$ for each agent i yields an efficient partnership, that is, a^* is an equilibrium of the game induced by this sharing rule.

3. There are 2 agents. Let $A_i = [0; 2]$, $y(a) = a_1 + a2$ and $c_i(a_i) = \frac{1}{2}a_i^2$.

   (a) Show that the efficient action profile is $a^* = (1, 1)$. Using your answer to 1(c) above, decide whether or not this environment admits an efficient partnership.

   (b) Suppose that agent 2 plays the pure strategy a_2^*, whereas agent 1 plays 0 and 2 with equal probability of $\frac{1}{2}\epsilon$, where $\epsilon > 0$ is small, and plays 1 with probability $1 - \epsilon$. Call this mixed strategy σ_1^ϵ (Notice that σ_1^ϵ approaches a_1^* as $\epsilon \to 0$)

      i. If agent 2 plays a_2^* then agent 1 can generate the interval [1,3] of output with behavior in A_1. Show that the sharing rule given by $S_1(y) = (y - 1)^2/2$ and $S_2(y) = y - S_1(y)$ for all $y \in [1; 3]$ renders σ_1^ϵ a best response to a_2^*

      ii. Assuming that agent 1 plays σ_1^ϵ, find budget balanced payments for output outside the interval [1; 3] that render a_2^* a best response to σ_1^ϵ

197 Rahman

Question IV.2 Spring 2015 majors

Consider the Prisoners' Dilemma with imperfect public monitoring.

	C	D
C	1,1	-1,2
D	2,-1	0,0

	C	D
C	p_2	p_1
D	p_1	p_0

Every period, players observe a binary signal $\omega \in \Omega = \{g, b\}$ that is IID conditional on players' actions. The bi-matrix on the left describes flow payoffs, and the matrix on the right describes the conditional probability of g, where $0 < p_0 < p_1 < p_2 < 1$. Players share a common discount factor $\delta < 1$. Assume that players have access to a public randomization device, and consider the following strategy profile. Both players begin in a cooperative regime, which calls for cooperation. If the signal is g then they remain in the cooperative regime. If the signal is b then with some probabilityα both players enter a punishment regime, which calls for defection, and remain in the punishment regime henceforth. With the remaining probability, $1 - \alpha$, players continue in the cooperative regime.

1. Find the maximal strongly symmetric pure-strategy equilibrium payoffs for every$\delta \in [0, 1)$. Under what conditions on the parameters of the problem is this feasible?

2. Now suppose that, instead of the public signal being observed every period, players do not see anything during each entire T-period block of time, where $T \in \mathbb{N}$, but all the T signals from the block are observed by players at the end of each block. Consider the following strategy profile. Players begin in a cooperative regime, which calls for cooperation throughout the block. If the history of signals throughout the block is not b every period, then players remain in the cooperative regime. Otherwise, with some probability α both players enter a punishment regime, which calls for defection, and remain in the punishment regime henceforth. With the remaining probability, $1 - \alpha$, players continue in the cooperative regime for the next block. Redo part 1 above under this assumption. Discuss whether or not and the extent to which cooperation can be sustained as $\delta \to 1$ and T can vary arbitrarily.

3. Let $\delta = e^{-r\Delta t}$, where $r > 0$ is a discount rate and $\Delta t > 0$ is the length of time between interactions. Given x such that $x_0 < x_1 < x_2$, let $p_k = \frac{1}{2}[1 + x_k\sqrt{\Delta t}]$ for each k, and assume that Δt is small enough that $0 < p_k < 1$ for all k. Finally, let the block length T depend on Δt according to $T(\Delta t) = c/\Delta t$ for some constant length of calendar time $c > 0$. (Ignore integer constraints.) Using the strategies of part 2 above, compare equilibrium payoffs as $r \to 0$ versus$\Delta t \to 0$.

198 Werner

Question I.1 Fall 2015 majors

Consider demand function $d : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_{++}$ of prices and income satisfying budget equation $pd(p, w) = w$ for every p and w.

(a) Show that if d is a Walrasian demand function of a consumer with locally non-satiated utility function, then the Generalized Weak Axiom of Revealed Preference (GWARP) holds for every T-tuple of price-quantity pairs $\{p^t, x^t\}_{t=1}^T$ where $x^t = d(p^t, w^t)$, $p^t \in \mathbb{R}^L_{++}$ and $w^t \in \mathbb{R}_+$ for every $t = 1, \ldots T$. State GWARP.

(b) Consider the following example of demand function for $L = 2$:

$$\hat{d}(p, w) = \begin{cases} (\frac{w}{p_1}, 0) & \text{if} p_1 \geq p_2 \\ (\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}) & \text{if} p_1 < p_2 \end{cases}$$

Show that GWARP does not hold for $\hat{d}$.

(c) State the Afriat's Theorem. The proof is not required.

199 Werner

Question I.2 Fall 2015 majors

Consider two real-valued random variables $\hat{y}$ and $\hat{z}$ on some state space (i.e. probability space) with $E(\hat{y}) = E(\hat{z})$; where E denotes expected value.

(i) State a definition of $\hat{y}$ being (weakly) more risky than $\hat{z}$ in terms of cumulative distribution functions. Prove that if $\hat{y}$ is more risky than $\hat{z}$, then

$$E[v(\hat{z})] \geq E[v(\hat{y})]$$

for every non-decreasing, continuous and concave utility function $v : \mathbb{R} \to \mathbb{R}$: You may assume in your proof that v is (twice) differentiable.

(ii) Give an example of two random variables $\hat{y}$ and $\hat{z}$, $\hat{y} \neq \hat{z}$, with the same expected values such that neither $\hat{y}$ is more risky than $\hat{z}$ nor $\hat{z}$ is more risky than $\hat{y}$. Justify your answer.

(iii) Show that $\lambda\hat{z}$ is more risky than $\hat{z}$ for every $\lambda \geq 1$ and every $\hat{z}$ with $E(\hat{z}) = 0$. Here, you may use without proof the fact that the necessary condition for more risky from part (i) is also sufficient.

200 Allen

Question II.1 Fall 2015 majors

This question concerns the first welfare theorem in a pure exchange economy with l commodities and n agents (indexed by the subscripts $i = 1, 2, \ldots n$), where each agent i has consumption set $\mathbb{R}^L_+$, initial endowment $e_i \in \mathbb{R}^L_{++}$ and preferences $\succeq_i$ defined on $\mathbb{R}^L_+$ which are assumed to be complete continuous preorders.

(a) For this economy, state the first welfare theorem. Be sure to state any additional assumptions that are needed and to define any technical terms that appear in your additional assumptions (if any).

(b) Define competitive equilibrium for this economy.

(c) Define the sets of weakly and strongly Pareto optimal allocations for this economy.

(d) Prove the theorem that you stated in part (a).

(e) When economists discuss market failure, they generally refer to situations or conditions that can cause the conclusion of the first welfare theorem to fail to hold. These include the following:

(i) Externalities
(ii) Strategic behavior (or "market power")
(iii) Uncertainty and incomplete or asymmetric information

For each of these (considered separately), explain clearly where your statement of the first welfare theorem (implicitly or explicitly) excludes these cases. [It might help to consider how each of (i), (ii), and (iii) would change your definitions of a pure exchange economy, competitive equilibrium, and Pareto optimality.]

201 Allen

Question II.2 Fall 2015 majors

It has been claimed - for instance in the Arrow and Hahn book on general equilibrium theory, although this is not an exact quote - that the problem with nonconvexities is that they cause discontinuities in (aggregate) demand which prevent one from proving the existence of competitive equilibrium.

(a) For a pure exchange economy with ‘ commodities and n traders (indexed by $i = 1, 2, \ldots n$) each having consumption sets $\mathbb{R}^L_+$, initial endowment vectors $e_i \in \mathbb{R}^L_{++}$, and preferences $\succeq_i$ defined on $\mathbb{R}^L_+$ which are assumed to be complete continuous preorders which are strictly monotone but not necessarily convex, evaluate the accuracy of this claim. In particular, is it true or false? Why? Please explain your answer clearly and carefully.

(b) Would it make a difierence in part (a) if the word "convex" were replaced by "strictly (or strongly) convex"? Explain briefly.

(c) Would it make a difierence in part (a) if the word "convex" were replaced by "weakly convex"? Explain briefly.

(d) Suppose that we modify the economy to one in which all of the preferences are convex when defined on $\mathbb{R}^L_+$, but each agent i's consumption set is restricted to a subset X_i of $\mathbb{R}^L_+$ where X_i is closed but not necessarily convex. Does this change your answer to part (a)? If so, how? Explain your answer clearly and carefully.

(e) For the economy in part (d), define competitive equilibrium.

(f) Now suppose that instead of n agents, the economy has uncountably many agents. Does this change your answers to parts (a) and (d)? Explain. Your answer should state any additional assumptions that are needed. It might be helpful to define competitive equilibrium in this context.

202 Rustichini

Question III.1 Fall 2015 majors

(a) Prove or disprove: an extensive form game of perfect recall for every player is a linear game.

(b) Consider an extensive form game of perfect information with a countable set of nodes, not necessarily finite. Define an extension of the concept of backward induction for this game. Does it define an equilibrium in pure strategies?

(c) Prove or disprove: for every finite extensive form game there exists an equivalent finite extensive form game in which nature moves once and only once, at the initial node of the new game. Provide a precise definition of 'equivalent game' and use it in your answer.

(d) Write the formulas giving a mixed strategy induced by a behavioral strategy, and a behavioral strategy induced by a mixed strategy. Illustrate with examples why the induced strategies may not be unique (or prove that they are).

203 Rustichini

Question III.2 Fall 2015 majors

(a) Define subgame perfect equilibrium for an extensive form game, not necessarily finite.

(b) A stochastic game over a countable set of times T(; : : : ;1g is defined as follows:

- A finite state space $k \in Kf \equiv \{1, \ldots, m\}$;
- A finite set of players $i \in I \equiv \{1, \ldots, n\}$ a finite action set A^i for every player, $A = \times A^i$, a utility function giving $u^i(a; k)$ for every i, a, k. Final utilities are discounted at the same discount for all players
- A transition function $(k, a) \in K \times A \to P(\cdot|k, a) \in \Delta(K)$
- Players are informed of the actions profile in the past period and the state realization in the current period before their simultaneous choice of action

(c) Prove that a stochastic game has a subgame perfect equilibrium

(d) Find a non-Markov subgame perfect equilibrium of the stochastic game with $m = n = 2$, the transition function does not depend on a, and

$$\forall k \quad P(k, k) = p > 0.5$$

and the following utility fucntions: for $k = 1$,

	L	R
T	4,1	0,0
M	0,0	1,4

for $k = 2$

	L	R
T	2,2	0,3
M	3,0	1,1

204 Rahman

Question IV.1 Fall 2015 majors

On the Relevance of Private Information
Consider an economy with two agents whose utility is quasi-linear with respect to some money commodity, as usual. There is also an indivisible public good that can either be provided or not. The cost of public good provision is $c > 0$. If the public good is provided with probability p then the agents must jointly pay the amount pc ex ante. Suppose that each agent i is equally likely to be one of two possible types, $t_i \in \{L, H\}$. Let $v_i(L) = 0$ be the utility to each agent i from the public good if i's type is L and $v_i(H) = 1$ if i's type is H. Each agent's utility from the public good not being provided is 0, regardless of players' types. Participation in the public good provision problem is voluntary: agents have an (interim) outside option whose worth is normalized to 0, regardless of type, if they choose not to participate. Type profiles are drawn from a common prior distribution. Each player i's beliefs regarding the other player's type are posterior probabilities derived from i's own type realization using Bayes' Rule and the common prior.

1. Suppose that types are independently distributed, so both agents believe that the other player is equally likely to be of either type.
   (a) Find an ex ante budget-balanced mechanism that renders the efficient allocation incentive compatible and individually rational when $c \leq 2/3$.
   (b) Find an ex ante budget-balanced mechanism that renders the efficient allocation incentive compatible when $c > 1$.
   (c) Show that no such mechanism exists when $2/3 < c < 1$
2. Suppose that $c < 1$ and types are correlated. Specifically, each agent believes that the other agent is of the same type as him or herself with probability 2/3 and the other type with probability 1/3. Consider the following mechanism: the public good is provided if and only if at least one agent reports H, and monetary payments are made by each player according to the table of reported types below.

	L	H
L	1/2-c/3, 1/2-c/3	2c/3-1/2, 1/2+2c/3
H	1/2+2c/3, 2c/3-1/2	2c/3-1/2, 2c/3-1/2

Report-Contingent Payments

   (a) Show that this mechanism is ex post efficient, ex ante budget balanced and induces truthful reporting as a dominant strategy, but not individually rational.
   (b) Now add to the previous mechanism the following lottery: for each agent i, if the other agent reports L then i receives $\frac{3}{2} - \frac{2c}{3}$, if the other agent reports H then i pays $\frac{5}{2} - \frac{4c}{3}$. Show that this new mechanism fulfills all the desiderata of part (a) plus individual rationality. Does it extract all the surplus?

205 Rahman

Question IV.2 Fall 2015 majors

The Limits of Price Discrimination
Consider an economy populated by a continuum of buyers, of mass one, with single-unit demands for some good. A buyer's valuation of the good, or maximum willingness to pay, is either 1, 2 or 3. A *market* is any vector belonging to $\Delta = \{x \in \mathbb{R}^3_+ \sum_i x_i = 1\}$, describing the mass of buyers with each of the three possible valuations. A monopolist produces the good in question at no cost.

(a) Let X_i be the subset of markets with the property that setting a price of i is profit-maximizing for the monopolist. Completely characterize the sets $X_1, X_2 and X_3$, and argue whether or not any other price can be optimal for the monopolist.

A *market segmentation* of a given market x^0 is a compound lottery that integrates to x^0: an n-tuple of markets $x = (x^1, \dots, x^n)$ and masses $q = (q^1, \dots, q^n)$ for some $n \in N$ such that $q^1 + \dots + q^n = 1$ and

$$\sum_{i=1}^{n} q^i x^i_j = x^0_j \quad \forall j \in \{1, 2, 3\}$$

Intuitively, the market x^0 is segmented into n market segments, the jth market segment has a mass q^j of buyers and its demographic composition is x^j . Given a market segmentation (x; q), assume that the monopolist can charge different prices in different market segments, but otherwise cannot discriminate within markets. Suppose that the market is actually $x^0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2. Without market segmentation, calculate both consumer and producer surplus assuming that the monopolist behaves optimally.

3. Find a market segmentation and pricing strategy for the monopolist such that every consumer is served yet the monopolist extracts all the surplus from the buyers.

4. Find a market segmentation and pricing strategy for the monopolist such that producer surplus is the same as in part 2 but consumer surplus is zero.

5. Find a market segmentation and pricing strategy for the monopolist such that producer surplus as the same as in part 2 but total surplus (that is, producer plus consumer surplus) is the same as in part 3.

6. Discuss

206 Werner

Question I.1 Fall 2015 minors

Consider a consumer whose utility function u on the consumption set RL+ is strictly increasing and continuous. Let $x(p, w)$ be the Walrasian demand for price vector $p \in \mathbb{R}^L_{++}$ and income $w \in \mathbb{R}_{++}$: Assume that x is a differentiable function of prices and income.

State a denition of the Slutsky matrix of demand function x: Show that the Slutsky matrix is negative semi-definite. You may use properties of the Hicksian demand (including its relations to Walrasian demand) without proofs, but you need to state them clearly.

207 Werner

Question I.2 Fall 2015 minors

Consider an agent whose preferences under uncertainty have expected utility representation with the von Neumann-Morgenstern utility function

$$v(x) = \log(x)$$

for x ¿ 0:

(i) Consider gamble $\hat{z}$ with two possible outcomes: gain g or loss $-l$; with equal probabilities, for some $g, l > 0$: Show that - regardless of the expected value of the gamble - if the agent rejects taking gamble $\hat{z}$ at w (that is, she strictly prefers w to $w + \hat{z}$), then she will also reject taking any bigger-scale gamble $t\hat{z}$ at w; for any $t \geq 1$.

(ii) Suppose that $g = l$. Find the risk compensation $\rho(w, \hat{z})$ as an explicit function of g and deterministic wealth w.

(iii) Suppose that there are two assets: a risk-free asset with (per-dollar, or gross) return $\hat{r}$ and a risky asset with return $\bar{r}$ which may take either one of two values, r_1 or r_2, with equal probabilities. Suppose that $r_1 > \hat{r} > r_2$: Find this agent's optimal investment in the risky asset as a function of initial wealth w and returns $\hat{r}$ and r_1, r_2. Verify whether the optimal investment is an increasing function of wealth and a decreasing function of risk-free return $\hat{r}$.

208 Allen

Question II.1 Fall 2015 minors

This question concerns the first welfare theorem in a pure exchange economy with l commodities and n agents (indexed by the subscripts $i = 1, 2, \ldots n$), where each agent i has consumption set $\mathbb{R}^L_+$, initial endowment $e_i \in \mathbb{R}^L_{++}$ and preferences $\succeq_i$ defined on $\mathbb{R}^L_+$ which are assumed to be complete continuous preorders.

(a) For this economy, state the first welfare theorem. Be sure to state any additional assumptions that are needed and to define any technical terms that appear in your additional assumptions (if any).

(b) Define competitive equilibrium for this economy.

(c) Define the sets of weakly and strongly Pareto optimal allocations for this economy.

(d) Prove the theorem that you stated in part (a).

(e) When economists discuss market failure, they generally refer to situations or conditions that can cause the conclusion of the first welfare theorem to fail to hold. These include the following:

(i) Externalities
(ii) Strategic behavior (or ”market power”)
(iii) Uncertainty and incomplete or asymmetric information

For each of these (considered separately), explain clearly where your statement of the first welfare theorem (implicitly or explicitly) excludes these cases. [It might help to consider how each of (i), (ii), and (iii) would change your definitions of a pure exchange economy, competitive equilibrium, and Pareto optimality.]

209 Allen

Question II.2 Fall 2015 minors

This question concerns the microeconomic implications of uncertainty (possibly combined with incomplete or asymmetric information) for perfectly competitive markets. To simplify, assume that there are two states of the world, denoted H (for heads) and T (for tails).

(a) Write down a simple model of a (general equilibrium) pure exchange economy with uncertainty. Be sure to define all notation you introduce.

(b) Define competitive equilibrium in your model. Be sure to state and discuss any assumptions you make, including assumptions about markets, information, timing, etc. [There may be more than one correct answer here, depending on your model in part (a).]

(c) Compare and contrast the properties of competitive equilibrium with and without uncertainty. Include a brief discussion of any differences in existence results and welfare theorems.

210 Rustichini

Question III.1 Fall 2015 minors

(a) Define subgame perfect equilibrium for definite extensive form games. You do not needto define an extensive form game. Illustrate the definition with:

(i) An example of a Nash equilibrium which is not subgame perfect.

(ii) An example of a subgame perfect equilibrium which cannot be obtained by backward induction.

(b) Prove that a subgame perfect equilibrium induces a subgame perfect equilibrium in every subgame of the original game.

(c) Consider a subgame G' of a game G, starting at a node x of G. Let β profile of behavioral strategies, which is a Nash equilibrium of G' . Consider the truncated game obtained by replacing in G the subgame G' with a terminal node, and assign to this node the payoff induced by β, and let also γ be an equilibrium of such truncated game. Finally consider the behavioral strategy profile in G that is given by γ in all the nodes in G that are not in G' , and by β in G' . Prove that this behavioral strategy profile is a Nash equilibrium of G.

211 Rustichini

Question III.2 Fall 2015 minors

(a) Define a normal form game and its mixed extension.

(b) Give an interpretation of payoffs in the mixed strategy extension.

(c) Define two games to be equivalent if they have the same set of players, the sameaction set for each player, and the same best response correspondence for every player. What is the class of transformations of the payoffs in a normal form game that give an equivalent game? Prove your answer.

(d) Illustrate the concept of correlated equilibrium with an example of a correlated equilibrium that is not a Nash equilibrium.

(e) Show that the set of correlated equilibrium payoff outcomes is a convex set. Is there a game with a set of Nash equilibrium payoff outcomes which is convex, not a singleton and not equal to the set of all payoff outcomes?

212 Rahman

Question IV.1 Fall 2015 minors

On the Relevance of Private Information

Consider an economy with two agents whose utility is quasi-linear with respect to some money commodity, as usual. There is also an indivisible public good that can either be provided or not. The cost of public good provision is $c > 0$. If the public good is provided with probability p then the agents must jointly pay the amount pc ex ante. Suppose that each agent i is equally likely to be one of two possible types, $t_i \in \{L, H\}$. Let $v_i(L) = 0$ be the utility to each agent i from the public good if i's type is L and $v_i(H) = 1$ if i's type is H. Each agent's utility from the public good not being provided is 0, regardless of players' types. Participation in the public good provision problem is voluntary: agents have an (interim) outside option whose worth is normalized to 0, regardless of type, if they choose not to participate. Type profiles are drawn from a common prior distribution. Each player i's beliefs regarding the other player's type are posterior probabilities derived from i's own type realization using Bayes' Rule and the common prior.

1. Suppose that types are independently distributed, so both agents believe that the other player is equally likely to be of either type.
   (a) Find an ex ante budget-balanced mechanism that renders the efficient allocation incentive compatible and individually rational when $c \leq 2/3$.
   (b) Find an ex ante budget-balanced mechanism that renders the efficient allocation incentive compatible when $c > 1$.
   (c) Show that no such mechanism exists when $2/3 < c < 1$
2. Suppose that $c < 1$ and types are correlated. Specifically, each agent believes that the other agent is of the same type as him or herself with probability 2/3 and the other type with probability 1/3. Consider the following mechanism: the public good is provided if and only if at least one agent reports H, and monetary payments are made by each player according to the table of reported types below.

	L	H
L	1/2-c/3, 1/2-c/3	2c/3-1/2, 1/2+2c/3
H	1/2+2c/3, 2c/3-1/2	2c/3-1/2, 2c/3-1/2

Report-Contingent Payments

   (a) Show that this mechanism is ex post efficient, ex ante budget balanced and induces truthful reporting as a dominant strategy, but not individually rational.
   (b) Now add to the previous mechanism the following lottery: for each agent i, if the other agent reports L then i receives $\frac{3}{2} - \frac{2c}{3}$, if the other agent reports H then i pays $\frac{5}{2} - \frac{4c}{3}$. Show that this new mechanism fulfills all the desiderata of part (a) plus individual rationality. Does it extract all the surplus?

213 Rahman

Question IV.2 Fall 2015 minors

The Limits of Price Discrimination
Consider an economy populated by a continuum of buyers, of mass one, with single-unit demands for some good. A buyer's valuation of the good, or maximum willingness to pay, is either 1, 2 or 3. A *market* is any vector belonging to $\Delta = \{x \in \mathbb{R}^3_+ \sum_i x_i = 1\}$, describing the mass of buyers with each of the three possible valuations. A monopolist produces the good in question at no cost.

(a) Let X_i be the subset of markets with the property that setting a price of i is profit-maximizing for the monopolist. Completely characterize the sets $X_1, X_2 and X_3$, and argue whether or not any other price can be optimal for the monopolist.

A *market segmentation* of a given market x^0 is a compound lottery that integrates to x^0: an n-tuple of markets $x = (x^1, \ldots, x^n)$ and masses $q = (q^1, \ldots, q^n)$ for some $n \in N$ such that $q^1 + \cdots + q^n = 1$ and

$$\sum_{i=1}^{n} q^i x^i_j = x^0_j \quad \forall j \in \{1, 2, 3\}$$

Intuitively, the market x^0 is segmented into n market segments, the jth market segment has a mass q^j of buyers and its demographic composition is x^j . Given a market segmentation (x; q), assume that the monopolist can charge different prices in different market segments, but otherwise cannot discriminate within markets. Suppose that the market is actually $x^0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2. Without market segmentation, calculate both consumer and producer surplus assuming that the monopolist behaves optimally.

3. Find a market segmentation and pricing strategy for the monopolist such that every consumer is served yet the monopolist extracts all the surplus from the buyers.

4. Find a market segmentation and pricing strategy for the monopolist such that producer surplus is the same as in part 2 but consumer surplus is zero.

5. Find a market segmentation and pricing strategy for the monopolist such that producer surplus as the same as in part 2 but total surplus (that is, producer plus consumer surplus) is the same as in part 3.

6. Discuss

214 Werner

Question I.1 Spring 2016 majors

Consider a firm with strictly increasing production function $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ of two inputs.

(a) State a definition of function f being supermodular. Give an example of a supermodular production function other than the linear function. Justify your answer.

(b) Let x_i^* denote the profit maximizing demand for input i, for $i = 1, 2$ Show that, if f is supermodular, then x_i^* is monotone nonincreasing in the price w_1 of input 1, for $i = 1, 2$. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly. You may assume that x^* is single valued.

(c) Suppose that the quantity of input 2 is held fixed at $\bar{x}_2 > 0$. and consider the "short run" profit maximization problem. Let $x_1^8(w_1, \bar{x}_2)$ denote the short-run profit maximizing demand for input 1, assumed uniquely defined. Show that x_1^* is monotone nonincreasing in w_1

(d) Show that if f is supermodular, then x_1^s is monotone nondecreasing in $\bar{x}_2$?

(e) Use (a-d) to show that the following LeChatelier principle holds:

$$x_1^s(w_1, x_2^*(w_1)) \leq x_1^s(w_1, x_2^*(w_1')) \leq x_1^s(w_1', x_2^*(w_1'))$$

for $w_1' \leq w_1$

215 Werner

Question I.2 Spring 2016 majors

Consider preference relation $\succeq$ defined on the set of non-negative contingent claims $\mathbb{R}^S_+$ on a state-space with S states of nature. Assume that $\succeq$ is continuous and strictly increasing.

(a) State necessary and sufficient condition(s) for $\succeq$ to have state-separable utility representation when $S \geq 3$

(b) Prove that the condition you stated in (a) is indeed necessary.

(c) Does the condition you stated in (a) remain necessary and sufficient in the case of two states, i.e., $S = 2$? Justify your answer.

(d) Describe the Ellsberg paradox. Show that the typical pattern of preferences among bets in the Ellsberg paradox violates the condition for state-separable utility representation you stated in (a).

216 Allen

Question II.1 Spring 2016 majors

This question concerns competitive equilibrium in pure exchange economies. To begin, consider an economy with l commodities and N traders, indexed by $i = 1, 2, \ldots, N$, each having consumption set $\mathbb{R}^l_+$, initial endowment vector $e_i \in \mathbb{R}^l_{++}$, and preferences $\precsim_i$ which are assumed to be complete preorders on $\mathbb{R}^l_+$

(a) Define competitive equilibrium for this economy using this notation.

(b) State a set of additional assumptions that suffice to guarantee that the economy has a competitive equilibrium. In fact, state a theorem on the existence of competitive equilibrium for the economy with your additional assumptions.

(c) Sketch a proof of your theorem.

(d) For each additional assumption, briefly state its economic interpretation and significance for the proof of your theorem.

(e) Now suppose that, instead of some finite number l (integer) of commodities, your economy has infinitely many commodities (and consumption sets are no longer $\mathbb{R}^l_+$, etc.). Discuss the interpretation of such models and the economic motivations for the study of such models.

(f) There can be many different infinite dimensional commodity spaces. Discuss the modeling choice among them, being clear to distinguish between technical considerations and issues involving the nature of commodities in the economy and the economic questions to be analyzed using the model.

217 Allen

Question II.2 Spring 2016 majors

This question concerns the first welfare theorem in a pure exchange economy with l commodities and N agents, indexed by $i = 1, 2, \ldots, N$, each having consumption set $\mathbb{R}^l_+$, initial endowment vector $e_i \in \mathbb{R}^l_{++}$, and preferences $\precsim_i$ which are assumed to be complete preorders on $\mathbb{R}^l_+$.

(a) Define the sets of weakly and strongly Pareto optimal allocations.

(b) For allocations (vectors in $\mathbb{R}^{IN}_+$) in this economy, define weakly and strongly Pareto dominance.

(c) What properties do they have as binary relations? Explain briefly.

(d) Define competitive equilibrium in this economy using notation.

(e) State the first welfare theorem.

(f) Prove the theorem you state in part (e).

(g) Does it matter for your theorem in part (e) whether preferences are strongly/strictly convex, convex, or weakly concave? Explain.

(h) Would it matter for your theorem if we were to replace the consumption sets $\mathbb{R}^l_+$ for each agent i by $X_i \subset \mathbb{R}^l_+$ where each X_i is a closed but not necessarily convex set? Explain.

218 Rustichini

Question III.1 Spring 2016 majors

(a) Define correlated equilibria of a normal form game G.

(b) Describe correlated equilibria as the Nash equilibria of a game in which before the play an action profile is chosen randomly by a third party, and each player is told his action in that action profile. Each player is free to choose any action after such recommendation. Note this defines an extensive form game (set payoffs of the third party equal to a constant if you want to consider him a player). Call G^e this extensive form game associated with G. Make clear the relation between the correlated equilibria and the equilibria of this game, which are said to implement the correlated equilibrium.

(c) Consider now the following game:

	L	R
T	$4,2$	$3,4$
B	$5,1$	$0,0$

- Describe all the correlated equilibria of the game in (c).
- Describe the game G^e if G is the game in (c).
- Describe the equilibria of G^e.

(d) A Nash equilibrium α of a normal form game is said to be a strict equilibrium if for every player i, and every action $a^i \in A^i$ such that $\alpha^i\left(a^i\right) = 0$

$$u^i\left(a^i, \alpha^{-i}\right) < u^i(\alpha)$$

(note, a strict inequality). A correlated equilibrium μ s called a strict correlated equilibrium if the strategy vector of G^e implementing μ is a strict equilibrium of G^e

- Does every game in normal form have a strict correlated equilibrium?
- What are the strict correlated equilibria, if any, of the game in (c)?

219 Rustichini

Question III.2 Spring 2016 majors

(a) Prove that every extensive form game with perfect recall has a subgame perfect equilibrium in mixed strategies. (b) Consider now the normal form game:

	L	R
T	$4,0$	$1,3$
B	$3,0$	$1,5$

(a) Suppose that players play this game twice; after the first round has been played, they are informed of the action of the other. The payoff after the two rounds are the sum of the payoff in every round. Describe this extensive form game, call it Γ, in detail.

(b) Describe all the subgames of Γ.

(c) Find all the subgame perfect equilibria of Γ.

220 Rahman

Question IV.1 Spring 2016 majors

A. Welfare and Rationing
A seller owns an object, and values it at 0. There is a buyer with valuation $v \sim U[0, 1]$. The seller does not know the buyer's valuation, and designs an optimal mechanism to fulfill some objective, whereby the seller asks for the buyer's valuation and then awards the object to the buyer with probability $q(v)$ and charges the buyer an amount of money $p(v)$ if the buyer reported a valuation v

(a) Assume that the seller wants to maximize own profit, $p(v)$

(a) Show that the seller's virtual surplus can be written as

$$2v - 1$$

(b) Describe the seller's optimal auction.

(b) Assume instead that the seller wants to maximize a weighted average of own profit, $p(v)$ (with weight $\alpha \in [0, 1]$), and consumer surplus, $v - p(v)$ (with weight $1 - \alpha$).

(a) Show that the seller's virtual surplus can be written as

$$(3\alpha - 1)v + 1 - 2\alpha$$

(b) Describe the seller's optimal auction as a function of $\alpha \in [0, 1]$

221 Rahman

Question IV.2 Spring 2016 majors

B. The Social Value of Public Information
There is a continuum of agents, uniformly distributed on $[0,1]$. Each agent $i \in [0,1]$ chooses $a_i \in R$. Let a be the action profile. Agent i has utility function

$$u_i(a,\theta) = -\left[(1-r)\left(a_i-\theta\right)^2 + r\left(L_i - \bar{L}\right)\right]$$

where $r \in (0,1)$ is a constant, θ represents the state of the economy,

$$L_i = \int_0^1 \left(a_j - a_i\right)^2 dj \quad \text{and} \quad \bar{L} = \int_0^1 L_j dj$$

Intuitively, agent i wants to minimize the distance between his action and the true state θ, and also minimize the distance between his action and the actions of others. The parameter r represents the trade-off between these two objectives. Social welfare (normalized) is

$$W(a,\theta) = \frac{1}{1-r}\int_0^1 u_i(a,\theta)di = -\int_0^1 \left(a_i-\theta\right)^2 di$$

Agent i forms expectations $E_i[\cdot] = E\left[\cdot|\mathcal{I}_i\right]$ conditional on his information $\mathcal{I}_i$ and maximizes expected utility.

(a) Show that each agent i 's optimal action is given by

$$a_i = (1-r)E_i[\theta] + rE_i[\bar{a}]$$

where $\pi = \int_0^1 a_j dj$ is the average action. Show that if θ is common knowledge then $a_i = \theta$ for every i is an equilibrium.

(b) Suppose that θ is drawn heuristically from a uniform prior over the real line. Agents observe a public signal

$$y = \theta + \eta$$

where $\eta \sim N\left(0,\sigma^2\right)$. Therefore, $\theta|y \sim N\left(y,\sigma^2\right)$. Now, agents maximize expected utility $E\left[u_i|y\right]$ given the same public information y. Show that $a_i(y) = y$ for every i is an equilibrium. Derive the following expression for welfare given θ :

$$E[W|\theta] = -\sigma^2$$

(c) Assume now that, in addition to the public signal, each agent i observes a private signal

$$x_i = \theta +$$

where $\epsilon_i \sim N\left(0,\tau^2\right)$ is (heuristically) independent across i and of θ and η. Let $\alpha = 1/\sigma^2$ and $\beta = 1/\tau^2$

Question IV.2 Spring 2016 majors cont.

(a) Show that

$$E_i[\theta] = E\left[\theta | x_i, y\right] = \frac{\alpha y + \beta x_i}{\alpha + \beta}$$

(b) Suppose that there is a number κ such that for every agent j

$$a_j\left(x_j, y\right) = \kappa x_j + (1-\kappa) y$$

Compute the value of $E_i[\bar{a}]$ and show that

$$\kappa = \frac{\beta(1-r)}{\alpha + \beta(1-r)}$$

defines an equilibrium.
(d) Show that expected welfare is given by

$$E[W(a,\theta)|\theta] = -\frac{\alpha + \beta(1-r)^2}{[\alpha + \beta(1-r)]^2}$$

Show that

$$\frac{\partial E[W|\theta]}{\partial \beta} > 0$$

and $\frac{\partial E[W|\theta]}{\partial \alpha} \geq 0$ if and only if $\frac{\beta}{\alpha} \leq \frac{1}{(2r-1)(1-r)}$ Interpret and compare with your answer to part (b).

222 Werner

Question I.1 Fall 2016 majors

Consider finite set of observations $\{(p^t, y^t)\}_{t=1}^T$ of pairs of price vectors and production plans of a firm, where $p^t \in R^L$ and $y^t \in R^L$.

(i) State the Weak Axiom of Profit Maximization (WAPM) for observations $\{(p^t, y^t)\}_{t=1}^T$ Give an example of two observations ($T = 2$) with two goods ($L = 2$; one input and one output) such that WAPM does not hold.

(ii) Prove that observations $\{(p^t, y^t)\}_{t=1}^T$ satisfy WAPM if and only if the convex production set given by

$$Y = \mathrm{co} f y^t : t = 1, \ldots, T\}$$

rationalizes these observations in the sense that y^t is a solution to profit maximization at p^t on Y for every t. Here, "co" denotes the convex hull.

(iii) Suppose that observations $\{(p^t, y^t)\}_{t=1}^T$ satisfy WAPM and that $p^t \geq 0$ for every t Show that the free-disposal production set $\hat{Y} = Y - R_+^L$ rationalizes observations $\{(p^t, y^t)\}_{t=1}^T$, too.

223 Werner

Question I.2 Fall 2016 majors

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values. You may assume that $\tilde{y}$ and $\tilde{z}$ take values in a finite interval $[a, b]$

(i) State a definition of $\tilde{z}$ second-order stochastically dominating (SSD) $\tilde{y}$.

(ii) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Provide a justification for why it is a sensible definition of more risky.

(iii) Suppose that $E(\tilde{z}) = E(\tilde{y})$ and that cumulative distribution functions F_y and F_z have the following weak single-crossing property: there exists $t^* \in R$ such that $F_y(t) - F_z(t) \geq 0$ for $t \leq t^*$ and $F_y(t) - F_z(t) \leq 0$ for $t \geq t^*$. Show that $\tilde{z}\,\mathrm{SSD}\,\tilde{y}$

224 Allen

Question II.1 Fall 2016 majors

One form of nonconvexities is generated by a production function that has set up costs, so that some strictly positive quantity of inputs is needed before a positive amount of output can be produced.

(a) For the case of a firm that uses a single input to produce a single output, draw an example of a production set for such a firm. Be sure to label your diagram.

(b) Would the first and second welfare theorems apply in such an economy? Explain your answers.

(c) Suppose that an economy with such a firm has a competitive equilibrium in which (each) such firm makes zero profits. Is this competitive equilibrium Pareto optimal? Why or why not?

(d) Microeconomic theory often argues that the problems that nonconvexities cause for the existence of competitive equilibrium can be solved by postulating the presence of a large number of economic agents (consumers or firms). Would it make sense to modify our basic 'model' for this situation by assuming that the economy contains either a very large finite number or infinitely many consumers and/or firms? If we were to change our 'model' in this way, would this restore the existence of at least approximate competitive equilibrium? Explain.

225 Allen

Question II.2 Fall 2016 majors

Consider a pure exchange economy with l commodities and n traders, indexed by subscripts $i = 1, 2, \dots n$; where each trader i has consumption set $\mathbb{R}^L_+$, initial endowment $e_i \in \mathbb{R}^L_{++}$, and preferencesi, which are assumed to be complete continuous preorders on $\mathbb{R}^L_+$.
(a) Define competitive equilibrium for this economy.

(b) Define the sets of weekly and strongly Pareto optimal allocations.

(c) State the second welfare theorem.

(d) State the first welfare theorem.

(e) Prove the theorem you stated in part (d).

(f) Assume $l \geq 2$, $n \geq 2$, and assume also that in this economy each ei is a strictly positive vector. Suppose that every feasible allocation in the economy is strongly Pareto optimal. What does this tell us about the properties that each preference relation must satisfy?

(g) In part (f), what does this tell us about the utility functions that can represent such preferences? In particular, can such preferences as in part (f) be represented by utilities that are not continuous?

(h) Find the set of all competitive equilibrium prices for an economy as in part (f).

(i) For an economy as in part (f), how many competitive equilibrium allocation vectors are associated with a single competitive equilbrium price vector?

226 Rustichini

Question III.1 Fall 2016 majors

Consider a preference order $\succeq$, and assume that it satisfies the von Neumann-Morgenstern (vNM) axioms. Let, for any two lotteries L and M, and any $\alpha \in [0, 1]$, (L, α, M) be the compound lottery that gives the lottery L with probability α and the lottery M with probability $1 - \alpha$

(a) State what a vNM representation is, and then state the vNM axioms in the form you prefer: the axioms you state must characterize preferences with the vNM representation.

(b) Prove that $\succeq$ satisfies the Sure Thing Principle (STP), namely that for any lotteries L, M, N and R and any $\alpha \in [0, 1]$

$$(L, \alpha, M) \succ (N, \alpha, M) \quad \text{if and only if} \quad (L, \alpha, R) \succ (N, \alpha, R)$$

If you assume the STP among your axioms, then prove that your axioms imply that the preference order has a vNM representation.

(c) Suppose that $L \succ M$; prove that for any $\alpha \in (0, 1]$

$$(L, \alpha, M) \succ M$$

(d) Prove that if u and v are two linear utility functions representing $\succeq$, then u is a positive affine transformation of v

227 Rustichini

Question III.2 Fall 2016 majors

Consider two normal form games. The first is BoS :

	l	r
T	$3,1$	$0,0$
B	$0,0$	$1,3$

The second is PD : 2

	l	r
T	$2,2$	$0,4$
B	$4,0$	$1,1$

Players play an extensive form game, for two periods. First they play BoS, player 1 moving first, and then player 2. Actions of the two players are revealed to both only after the move of player 2. In the next period the game is BoS again with probability 50 per cent, and PD with probability 50 per cent. Players are informed of which game they are playing, then player 1 chooses his action, player 2 chooses his, before he is told the action of player 1. Then the game ends and they are paid the sum of the payoffs in each game.

(a) Describe the extensive form game (EFG), drawing the game tree. You may let Nature move at nodes that are not the initial node.

(b) Is there an equivalent game where Nature moves at the initial node? Define precisely what you mean by equivalent in your statement.

(c) Define Subgame Perfect (SP) equilibria, and then describe at least three different SP equilibria of the EFG you defined in part (a).

(d) Find a constrained efficient SP equilibrium: that is a SP equilibrium such that there is not other SP equilibrium which gives a weakly larger payoff to both players and strictly larger to at leats one.

228 Rahman

Question IV.1 Fall 2016 majors

Global Games Consider a game with two players, 1 and 2, each of which has two actions, to either invest or not. If both players invest, they each get a payoff of $\theta \in R$. If a player invests and the other does not, the player who invested obtains a payoff of $\theta - 1$. A player who chooses not to invest obtains a payoff of 0 regardless of the other player's behavior. (a) Assume thatθ is common knowledge.

(i) Derive all the equilibria of this game as a function of θ

(ii) In a symmetric game, a symmetric pure-strategy profile (a, a) is said to risk dominate another pure-strategy profile (b, b) if, when a player believes that the other player will mix between a and b with probability 1/2 , the expected payoff from playing a exceeds that from playing b. Determine the risk dominant symmetric strategy profile as a function of θ.

(b) Now, assume instead that players do not observe θ. Heuristically, they believe thatθ is uniformly distributed over the real line. Instead of observing θ , each player i observes a signal

$$x_i = \theta + \epsilon_i$$

where $\epsilon_i \sim N(0; \sigma^2)$ and $(\epsilon_1; \epsilon_2)$ is a pair of jointly normal, IID random variables. Thus, a player i who observes x_i believes that $\theta | x_i \sim N(x_i; \sigma^2)$ and $x_j | x_i \sim N(x_i; 2\sigma^2)$.

(i) Find an equilibrium of this game.

(ii) Show that this equilibrium is the only strategy profile that survives iterated elimination of interim strictly dominated strategies.

(iii) How does this equilibrium depend onσ^2? Determine the limiting equilibrium behavior as $\sigma^2 \to 0$. Relate this limit to your answers on risk dominance above.

229 Rahman

Question IV.2 Fall 2016 majors

Insurance Markets Consider an insurance market with two types of consumers, high-risk types and low-risk types. High-risk types have a probability of accident given by $p_H = 1/2$, whereas for low-risk types it is $p_L = 1/3$. The utility of consumer i from wealth prospect $W = (W_1, W_2)$, where W_1 is wealth if no accident and W_2 is wealth in case of an accident, is given by

$$U_i(W) = \min_p \left\{ (1-p)W_1 + pW_2 : p_i - \frac{1}{6} \leq p \leq p_i + \frac{1}{6} \right\}$$

Each consumer is endowed with the initial wealth prospect $W_0 = (9, 3)$

(a) Derive the zero-profit line of insurance contracts for an insurance company that insures high-risk consumers only; call this line Z_H. Derive the zero-profit line of insurance contracts for an insurance company that insures low-risk consumers only; call this line Z_L. Derive the best insurance contract for high-risk types on Z_H; call this contract α_H

(b) Sketch an indifference curve for high-risk types through their endowment. (Hint: Calculate U_i separately under the cases $W_1 > W_2, W_1 < W_2$, and $W_1 = W_2$.) Sketch an indifference curve for high-risk types through α_H. Derive the contract on Z_L that gives high-risk types as much utility as α_H; call this contract α_L

(c) Sketch the indifference curve of low-risk types through α_L. Calculate the utility to low-risk types from α_L. Derive a necessary bound on the proportion of high-risk types in the economy for (α_L, α_H) to be a separating equilibrium.

230 Werner

Question I.1 Fall 2016 minors

Let $d : \mathcal{R}^L_{++} \times \mathcal{R}_+ \to \mathcal{R}^L_+$ be a demand function of prices and income satisfying budget equation $pd(p, w) = w$ for every p and w (i) Show that if d is a Walrasian demand function of a consumer with strictly increasing utility function, then the Generalized Weak Axiom of Revealed Preference (GWARP) holds for every T -tuple of price-quantity pairs $\{p^t, x^t\}_{t=1}^T$, where $x^t = d\left(p^t, w^t\right)$ $p^t \in \mathbb{R}^L_{++}$ and $w^t \in \mathcal{R}_+$ for every $t = 1, \ldots, T$. State GWARP.

(ii) Consider the following demand function for $L = 2$:

$$\hat{d}(p, w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } p_1 \geq p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if } p_2 > p_1 \end{cases}$$

Show that GWARP does not hold for $\hat{d}$.

(iii) State the Afriat's Theorem. The proof is not required.

231 Werner

Question I.2 Fall 2016 minors

There are two assets: a risk-free asset with (gross) return $\bar{r}$ and a risky asset with return r that takes value r_s in each state of nature $s \in S$. Consider an agent who allocates her initial wealth $w > 0$ between those two assets so as to maximize the expected utility of investment payoff. The agent is strictly risk averse. Further, the von Neumann-Morgenstern (or Bernulli) utility function v is strictly increasing and differentiable.

(i) Show that if the expected return on the risky asset $E(r)$ equals the risk free return $\bar{r}$ then the optimal investment in the risky asset is zero. Further, show that if $E(r) > \bar{r}$ then the optimal investment is strictly positive.

(ii) Suppose that the utility function v is $v(x) = -e^{-\alpha x}$ for some $\alpha > 0$. Show that the optimal investment in the risky asset does not depend on the agent's wealth w

232 Allen

Question II.1 Fall 2016 minors

One form of nonconvexities is generated by a production function that has set up costs, so that some strictly positive quantity of inputs is needed before a positive amount of output can be produced.

(a) For the case of a firm that uses a single input to produce a single output, draw an example of a production set for such a firm. Be sure to label your diagram.

(b) Would the first and second welfare theorems apply in such an economy? Explain your answers.

(c) Suppose that an economy with such a firm has a competitive equilibrium in which (each) such firm makes zero profits. Is this competitive equilibrium Pareto optimal? Why or why not?

233 Allen

Question II.2 Fall 2016 minors

Consider a pure exchange economy with l commodities and n traders, indexed by subscripts $i = 1, 2, \ldots, n$, where each trader i has consumption set $\mathbb{R}^l_+$, initial endowment $e_i \in \mathbb{R}^l_+$, and preferences $\preceq_i$, which are assumed to be complete continuous preorders on $\mathbb{R}^l_+$.

(a) Define competitive equilibrium for this economy.

(b) Define the sets of weekly and strongly Pareto optimal allocations.

(c) State the second welfare theorem.

(d) State the first welfare theorem.

(e) Prove the theorem you stated in part (d).

(f) Assume $l \geq 2, n \geq 2$, and assume also that in this economy each e_i is a strictly positive vector. Suppose that every feasible allocation in the economy is strongly Pareto optimal. What does this tell us about the properties that each preference relation must satisfy?

(g) In part (f), what does this tell us about the utility functions that can represent such preferences? In particular, can such preferences as in part (f) be represented by utilities that are not continuous?

234 Rustichini

Question III.1 Fall 2016 minors

Consider a preference order $\succeq$ that satisfies the von Neumann-Morgenstern axioms. Let, for any two lotteries L and M, and any $\alpha \in [0, 1], (L, \alpha, M)$ be the compound lottery that gives the lottery L with probability α and the lottery M with probability $1 - \alpha$. (a) Prove that $\succeq$ satisfies the Sure Thing Principle (STP), namely that for any lotteries L, M, N and R and any $\alpha \in [0, 1]$

$$(L, \alpha, M) \succ (N, \alpha, M) \text{ if and only if } (L, \alpha, R) \succ (N, \alpha, R)$$

If you assume the STP among your axioms, then prove that your axioms imply that the preference order has a vNM representation.

(b) Suppose that $L \succ M$; prove that for any $\alpha \in (0, 1]$

$$(L, \alpha, M) \succ M$$

235 Rustichini

Question III.2 Fall 2016 minors

Consider two normal form games. The first is BoS :

	l	r
T	$3,1$	$0,0$
B	$0,0$	$1,3$

The second is PD :

	l	r
T	$2,2$	$0,4$
B	$4,0$	$1,1$

Players play an extensive form game, for two periods. First they play BoS, player 1 moving first, and then player 2. Actions of the two players are revealed to both only after the move of player 2. In the next period the game is BoS again with probability 50 per cent, and PD with probability 50 per cent. Players are informed of which game they are playing, then player 1 chooses his action, player 2 chooses his, before he is told the action of player 1. Then the game ends and they are paid the sum of the payoffs in each game.

(a) Describe the extensive form of the game.

(b) Define Subgame Perfect (SP) equilibria.

(c) Describe at least three different SP equilibria of the game.

(d) Prove that the equilibria you describe are indeed SP.

236 Rahman

Question IV.1 Fall 2016 minors

Global Games Consider a game with two players, 1 and 2, each of which has two actions, to either invest or not. If both players invest, they each get a payoff of $\theta \in R$. If a player invests and the other does not, the player who invested obtains a payoff of $\theta - 1$. A player who chooses not to invest obtains a payoff of 0 regardless of the other player's behavior. (a) Assume thatθ is common knowledge.

(i) Derive all the equilibria of this game as a function of θ

(ii) In a symmetric game, a symmetric pure-strategy profile (a, a) is said to risk dominate another pure-strategy profile (b, b) if, when a player believes that the other player will mix between a and b with probability 1/2 , the expected payoff from playing a exceeds that from playing b. Determine the risk dominant symmetric strategy profile as a function of θ.

(b) Now, assume instead that players do not observe θ. Heuristically, they believe thatθ is uniformly distributed over the real line. Instead of observing θ , each player i observes a signal

$$x_i = \theta + \epsilon_i$$

where $\epsilon_i \sim N(0; \sigma^2)$ and $(\epsilon_1; \epsilon_2)$ is a pair of jointly normal, IID random variables. Thus, a player i who observes x_i believes that $\theta | x_i \sim N(x_i; \sigma^2)$ and $x_j | x_i \sim N(x_i; 2\sigma^2)$.

(i) Find an equilibrium of this game.((Hint: look for an equilibrium in cutoff strategies, that is, where each player chooses a threshold above which the player invests and below which the player does not. Calculate a player's expected payoff from investing assuming that the other player plays a cutoff strategy).

(ii) Show that this equilibrium is the only strategy profile that survives iterated elimination of interim strictly dominated strategies.(Hint start by finding thresholds of x above which it is suboptimal a dominant strategy to invest and below which it is a dominant strategy not to.)

(iii) How does this equilibrium depend onσ^2? Determine the limiting equilibrium behavior as $\sigma^2 \to 0$. Relate this limit to your answers on risk dominance above.

237 Rahman

Question IV.2 Fall 2016 minors

Insurance Markets Consider an insurance market with two types of consumers, high-risk types and low-risk types. High-risk types have a probability of accident given by $p_H = 1/2$, whereas for low-risk types it is $p_L = 1/3$. The utility of consumer i from wealth prospect $W = (W_1, W_2)$, where W_1 is wealth if no accident and W_2 is wealth in case of an accident, is given by

$$U_i(W) = \min_p \left\{ (1-p)W_1 + pW_2 : p_i - \frac{1}{6} \leq p \leq p_i + \frac{1}{6} \right\}$$

Each consumer is endowed with the initial wealth prospect $W_0 = (9, 3)$ (a) Derive the zero-profit line of insurance contracts for an insurance company that insures high-risk consumers only; call this line Z_H. Derive the zero-profit line of insurance contracts for an insurance company that insures low-risk consumers only; call this line Z_L. Derive the best insurance contract for high-risk types on Z_H; call this contract α_H (b) Sketch an indifference curve for high-risk types through their endowment. (Hint: Calculate U_i separately under the cases $W_1 > W_2, W_1 < W_2$, and $W_1 = W_2$.) Sketch an indifference curve for high-risk types through α_H. Derive the contract on Z_L that gives high-risk types as much utility as α_H; call this contract α_L (c) Sketch the indifference curve of low-risk types through α_L. Calculate the utility to low-risk types from α_L. Derive a necessary bound on the proportion of high-risk types in the economy for (α_L, α_H) to be a separating equilibrium.

238 Werner

Question I.1 Spring 2017 majors

Consider a lexicographic preference order on $\mathbb{R}^n_+$ with the first priority defined by a strictly increasing and continuous utility function $u(x)$ and the second priority being x_1, that is consumption of good 1.

(a) Show that this preference order is not continuous. Clearly state a definition of continuity.

(b) Suppose in addition that function u is strictly concave. Show that Walrasian demand for the lexicographic preference coincides with Walrasian demand for utility function u for every strictly positive price vector $p \in \mathbb{R}^n_{++}$ and income $w > 0$

(c) Give an example of a continuous and strictly increasing function u for $n = 2$ (two goods) that is concave but not strictly concave such that the demand for the lexicographic preference is different from the demand for u for some strictly positive prices and income. Justify your answer.

239 Werner

Question I.2 Spring 2017 majors

There are two assets: a risk-free asset with return r_f and a risky asset with return $\tilde{r}$ that may take one of two values r_L with probability $\pi > 0$ or r_H with probability $1 - \pi > 0$. All returns are per-dollar returns, that is, total returns for one dollar invested. Assume that $r_H > r_L$. An agent has expected utility function with von Neumann (or Bernoulli) utility index $v(x) = \ln(x)$, for $x > 0$, and initial wealth $w > 0$. Negative investment (i.e., short selling) is permitted for both assets.

(a) Show that there does not exist an optimal investment if $r_L \geq r_f$ or $r_H \leq r_f$

(b) Suppose that $r_H > r_f > r_L$. Find the optimal investment in the risky asset as a function of initial wealth w and returns r_f, r_H and r_L Consider now the following comparative statics exercise. The return $\tilde{r}$ is changed to a more risky return $\tilde{r}'$ that takes two possible values r'_H and r'_L with unchanged probabilities π and $1 - \pi$, respectively. The expected return on $\tilde{r}'$ is the same as on $\tilde{r}$.

(c) Derive necessary and sufficient condition for $\tilde{r}'$ being more risky than $\tilde{r}$ in terms of actual returns r_H, r_L, r'_H and r'_L. Prove your characterization.

(d) Is the optimal investment in the risky asset with more risky return $\tilde{r}'$ smaller then the optimal investment with less risky return $\tilde{r}$, everything else being unchanged? Justify your answer.

240 Allen

Question II.1 Spring 2017 majors

Consider a pure exchange economy with two goods and two consumers with $e_1 = e_2 = (1, 1)$ (a) Assume that preferences are represented by the utility functions $u_1(x, y) = \sqrt{xy}$ and $u_2(x, y) = 1$. Does the first fundamental theorem of welfare economics apply in this case? Why or why not? Are all competitive equilibria Pareto optimal? Explain your answer.

(b) Assume now that $u_2(x, y) = 2$ if $x + y \leq 2$ and $u_2(x, y) = \sqrt{xy}$ otherwise. Does the first welfare theorem apply here? Why or why not? Are all competitive equilibrium allocations Pareto optimal? Explain your answer.

241 Allen

Question II.2 Spring 2017 majors

Consider an exchange economy filled with n log-linear consumers $i = 1, \ldots, n$ having initial endowments $e_i = \lambda_i(1, \ldots, 1) = (\lambda_i, \ldots, \lambda_i) \in \mathbb{R}^\ell_{++}$ and utility functions $u_i : \mathbb{R}^\ell_{++} \to \mathbb{R}$ (defined on their consumption sets $\mathbb{R}^\ell_{++}$, so that there are ℓ commodities and strictly positive amounts must be consumed), defined by

$$u_i\left(x_i^1, \ldots, x_i^\ell\right) = \alpha_i^1 \log x_i^1 + \cdots + \alpha_i^\ell \log x_i^\ell$$

where each $\alpha_i^k (i = 1, \ldots, n, k = 1, \ldots, \ell)$ is a strictly positive scalar and $\sum_{k=1}^\ell \alpha_i^k = 1$ for all $i = 1, \ldots, n$ (a) Show that the aggregate excess demand of this economy could also be generated by a single Cobb-Douglas consumer. Specify the endowment and utility function for this consumer.

(b) Provide a more general condition on individual endowments that is sufficient for the aggregation result of part (a) to hold. Prove your result.

(c) Show that the conclusion in part (a) fails if the initial endowment vectors $e_i \in \mathbb{R}^\ell_{++}$ are arbitrary

242 Rustichini

Question III.1 Spring 2017 majors

(a) Define Correlated Equilibrium

(b) For the zero sum game below (entries indicate the payoff to the row player) find the value of the game, the optimal (equilibrium) strategies of the two players and the set of correlated equilibria.

	l	c	r
T	0	0	1
M	1	1	0
B	1	1	0

(c) Does every correlated equilibrium belong to the convex hull of the correlated strategies obtained as product of the optimal strategies?

(d) Prove that in any finite game (not necessarily zero sum) the correspondence defining the set of correlated equilibria is upper hemi-continuous in the utility of the players.

243 Rustichini

Question III.2 Spring 2017 majors

(a) Tic-Tac-Toe is played on a 3×3 grid, initially empty; two players move in sequence, the first marking a position with $\times$, and the second player with a $\bigcirc$; each player can put his mark in any position that is not occupied. The game ends if all positions are occupied; a player wins if he occupies first three adjoining squares, vertically, horizontally or diagonally. If no one does, the outcome is a tie. How many strategies does the first mover in Tic-Tac-Toe have? How many does the second player?

(b) Consider the N-Tic-Tac-Toe defined similarly, on an $N \times N$ grid. How many strategies does the first mover in Tic-Tac-Toe have? How many does the second player?

(c) Prove that in every two-player game with perfect information in which the outcomes are $\{$ Firstplayerwins, SecondPlayerwins, Draw$\}$, one and only one of the following three alternatives holds: (1) Player 1 has a winning strategy (2) player 2 has a winning strategy (3) Each of the two players has a strategy guaranteeing at least a draw.

244 Rahman

Question IV.1 Spring 2017 majors

Talent, Effort and Grades

(a) Consider two parties, an agent and the market. The agent chooses an unobservable action $a \in \mathbb{R}$ and incurs a private cost $c(a)$, where the function c is assumed to be differentiable. The market observes a performance variable $y \in \mathbb{R}$ that depends stochastically on effort and talent, denoted by $\theta \in \mathbb{R}$. Assuming that the agent's effort is $\hat{a}$, both the market and the agent attach a continuously differentiable probability density function $f(\theta, y|\hat{a}) > 0$ to the agent having talent θ and performance being y. The market rewards the agent its expectation of talent given performance and ostensible effort $\hat{a}$

$$E[\theta|y, a]$$

Before performance has realized, the agent reacts to the market's belief about effort, $\hat{a}$, by choosing an action a^* belonging to

$$\mathcal{B}(\hat{a}) = \arg\max_a E[E[\theta|y, \hat{a}]] - c(a)$$

where the expectation outside is with respect to performance and the one inside is with respect to talent. An equilibrium is a fixed point of $\mathcal{B}$. Assuming an interior solution, derive the following necessary condition for a^* to be an equilibrium:

$$\text{Cov}(\theta, \ell) = c'(a^*)$$

where (i) $\ell = \hat{f}_a/\hat{f}$, (ii) $\hat{f}$ is the conditional density of performance y given effort a, (iii) f_a is the partial derivative of $\hat{f}$ with respect to effort, and (iv) $c'(a^*)$ denotes the marginal cost of effort, i.e., the derivative of c

(b) Suppose that a student's grade is y, where depends on both talent θ and effort a, and $\theta \in \{0, 1, 5\}$; each of these three possible levels of talent is equally likely. Effort is binary: $a \in \{0, 1\}$. Assume that $c(0) = 0$ and $c(1) \in \left(\frac{1}{3}, \frac{2}{3}\right)$

(a) Assume full disclosure of the grade y to the market. Show that $a^* = 0$ is the only equilibrium. (Hint: Off path, assume that the market infers the correct effort to reconcile actual performance, but on path the market assumes that the agent's effort coincides with its original conjecture $\hat{a}$. Thus, if $\hat{a} = 1$ and $y = 5$, for instance, then the market infers that $a = 0$ and $\theta = 5$.)

(b) Now consider the following grade system: instead of observing y, the market observes only $z \in \{$ pass,fail $\}$ such that $z =$ pass if and only if $y \geq 1$. Show that now $a^* = 1$ is the unique equilibrium. Discuss.

245 Rahman

Question IV.2 Spring 2017 majors

Pure Common Values There are n potential bidders of a single unit of a good. Each bidder $i \in \{1, \ldots, n\}$ receives a signal $s_i \in S = [\underline{s}, \bar{s}]$ about the good's value. The signals s_i are IID across bidders from a continuously differentiable CDF $F(s_i)$ with density $f(s_i) > 0$. The bidders all assign the same value v to the good, given by

$$v(s_1, \ldots, s_n) = \max\{s_1, \ldots, s_n\}$$

The common value of the object is thus the maximum of the n independent signals. (This may be interpreted as a resale market where, after the auction, signals are publicly revealed.) A mechanism is a pair (q, t) such that $q_i(s) \geq 0$ and $\sum_i q_i(s) \leq 1$ and $t_i(s) \in \mathbb{R}$ for every bidder i and signal profile $s \in S^n$ where $q_i(s)$ is the probability that i gets the good and $t_i(s)$ is how much i pays the seller when reported signals are s. Every bidder has quasilinear utility:

$$u_i(s, q, t) = v(s)q_i(s) - t_i(s)$$

Let $U_i(s_i, s_i') = E[v(s)q_i(s_i', s_{-i})\,|s_i] - E[t_i(s_i', s_{-i})\,|s_i]$ and $U_i(s_i) = U_i(s_i, s_i)$ The mechanism (q, t) is incentive compatible if $U_i(s_i) \geq U_i(s_i, s_i')$; individually rational if $U_i(s_i) \geq 0$. Let

$$\widehat{Q}_{i,j}(x) = \int_{[\underline{\varepsilon}, x]^{n-1}} q_i(x, s_{-j})\, f_{-j}(s_{-j})\, ds_{-j}$$

be the probability, conditional on bidder j 's signal being x, that (i) the highest signal is x, and (ii) bidder i is allocated the good; write $\widehat{Q}_i(x) = \widehat{Q}_{i,i}(x)$. Let

$$\bar{Q}_i(x) = \sum_{j=1}^{n} \widehat{Q}_{i,j}(x)$$

be the total probability that bidder i is allocated the good and the highest signal is x. Finally, let $\hat{Q}(x) = \sum_i \widehat{Q}_i(x)$ and $\bar{Q}(x) = \sum_i \bar{Q}_i(x)$

(a) Show that if (q, t) is incentive compatible then

$$U_i(s_i) = U_i(\underline{g}) + \int_{\underline{\mu}}^{s_i} \widehat{Q}_i(x)dx$$

(b) Show that the total surplus T realized in the auction is given by

$$T = n \int_S x\bar{Q}(x)F^{n-1}(x)f(x)dx$$

(c) Show that if (q, t) is incentive compatible then its expected revenue is

$$R = \int_{\underline{z}}^{\bar{s}} \left[x\bar{Q}(x)nF^{n-1}(x) - \int_{\underline{z}}^{x} \widehat{Q}(y)dy\right] f(x)dx - \sum_{i=1}^{n} U_i(\underline{g})$$

Thus, if two incentive compatible mechanisms induce the same allocation and assign the same utilities to $\underline{s}$, they must generate the same expected revenue.

(d) Consider the following mechanism. A bidder with the highest signal is never allocated the good and its transfer is zero. (Ignore ties.) Each other bidder with a lower signal is allocated the good with probability $1/(n-1)$ and pays a $1/(n-1)$ share of the expectation of the highest of the $n-1$ other signals:

$$\hat{s} = \int_{\underline{s}}^{\bar{s}} x(n-1)F^{n-2}(x)f(x)dx$$

Question IV.2 Spring 2017 majors

(a) Show that this mechanism is individually rational.

(b) Show that the allocation induced by this mechanism maximizes the integrand in R above pointwise for each x. How much of the total surplus is kept by the seller?

(c) Show that, regardless of F, this mechanism fails to be incentive compatible. (Hint: Show that the highest type $\bar{s}$ benefits from pretending to be $\underline{\underline{g}}$)

246 Werner

Question I.1 Fall 2017 majors

Consider a preference relation $\succeq$ on the consumption set $X = \mathbb{R}_+^L$. Suppose that $\succeq$ is reflexive, transitive, complete, continuous and strictly inereasing (i.e., strongly monotone).

(a) State a definition of $\succeq$ having a utility representation. Is a utility representation, if it exists, unique? Explain. Preference relation $\succeq$ is said to be homothetic if the following holds for every $x, x' \in \mathbb{R}_+^L$ and every $\lambda > 0$: if $x \sim x'$ then $\lambda x \sim \lambda x'$

(b) Prove that $\succeq$ is homothetic if and only if it there exists a utility representation of $\succeq$ that is homogeneous of degree 1

(c) Is every utility representation of a homothetic preference relation homogeneous of degree 1? Justify your answer.

247 Werner

Question I.2 Fall 2017 majors

There are two assets: a risk-free asset with return r_f and a risky asset with return $\tilde{r}$. Returns are per-dollar returns, that is, total returns for one dollar invested. An agent has expected utility function with von Neumann (or Bernoulli) utility index $v(x) = -(\alpha - x)^2$, for $\alpha > 0$, and initial wealth $w > 0$ Assume that $\alpha > wr_f$. Negative investment (i.e., short selling) is permitted for both assets.

(a) Find the optimal investment in the risky asset as a function of the expected return and the variance of the return on the risky asset, the risk-free return, and the agent's wealth. Consider the following comparative statics exercise. The return $\tilde{r}$ is changed to a more risky return $\tilde{r}'$. The expected return on $\tilde{r}'$ is the same as on $\tilde{r}$, that is, $E(\tilde{r}') = E(\tilde{r})$

(b) State a definition of more risky return (or more risky random variable).

(c) Suppose that $E(\tilde{r}) > r_f$. Show that the optimal investment in the risky asset with more risky return $\tilde{r}'$ is smaller then the optimal investment with less risky return $\tilde{r}$, everything else being unchanged? Prove your answer.

248 Allen

Question II.1 Fall 2017 majors

Consider a pure exchange economy with $l \geq 2$ commodities and $n \geq 2$ consumers, indexed by subscripts $i = 1, \ldots, n$, where each consumer i has consumption set $\mathbb{R}^l_+$, initial endowment $e_i \in \mathbb{R}^l_+$, and preferences $\preceq_i$, which are assumed to be complete continuous preorders on $\mathbb{R}^l_+$.

(a) Define competitive equilibrium for this economy.

(b) State the second welfare theorem.

(c) State the first welfare theorem.

(d) Prove the theorem you stated in part (c).

Suppose that preferences are strongly monotone, strictly convex and homothetic, and that initial endowments are $e_i = \lambda_i(1, \ldots, 1) \in \mathbb{R}^\ell_{++}$ Further, assume that all agents have identical preferences.

(e) Show that competitive equilibrium allocation depends continuously on the endowent parameter $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{++}$

(f) Show that equilibrium prices do not depend on λ. Are competitive equilibrium prices necessarily unique?

(g) Do the results of part (f) hold if preferences are not identical (but satisfy all other conditions)? Justify.

249 Allen

Question II.2 Fall 2017 majors

Consider a two-agent exchange economy where traders have continuous concave utilities $u_i : \mathbb{R}^\ell_+ \to \mathbb{R}, i = 1, 2$. Let $e_1 \in \mathbb{R}^\ell_+$ and $e_2 \in \mathbb{R}^\ell_+$ with $e_1 + e_2 \neq 0$ be the endowments. Assume that u_1 and u_2 are strictly monotone with $u_1(0) = u_2(0) = 0$. The feasible utility set U is a set in $\mathbb{R}^2_+$ defined with a no-free disposal condition: $(\bar{u}_1, \bar{u}_2) \in U$ if and only if there exist $x_1 \in \mathbb{R}^\ell_+$ and $x_2 \in \mathbb{R}^\ell_+$ with $x_1 + x_2 = e_1 + e_2$ such that $u_1(x_1) = \bar{u}_1$ and $u_2(x_2) = \bar{u}_2$. Let $\tilde{U}$ denote the comprehensive hull of U, that is: $\tilde{U} \equiv U - \mathbb{R}^2_+$, or alternatively:

$$\tilde{U} = \left\{(\tilde{u}_1, \tilde{u}_2) \in \mathbb{R}^2 | \exists x_1, x_2 \in \mathbb{R}^\ell_+ \text{ with } x_1 + x_2 = e_1 + e_2\right.$$

such that $\tilde{u}_1 \leq u_1(x_1)$ and $\tilde{u}_2 \leq u_2(x_2)\}$

(a) Prove that U is a compact subset of $\mathbb{R}^2$.

(b) Is U convex?

(c) Prove that $\tilde{U}$ is closed.

(d) Prove that $\tilde{U}$ is convex.

(e) Let $\hat{U} = \left\{(\hat{u}_1, \hat{u}_2) \in \mathbb{R}^2 | \exists \lambda_1 \geq 0 \text{ and } \lambda_2 \geq 0 \text{ with } \lambda_1 + \lambda_2 = 1\right)$ such that. $\hat{u}_1 = u_1(\hat{x}_1), \hat{u}_2 = u_2(\hat{x}_2)$ with $(\hat{x}_1, \hat{x}_2) \in \arg\max\left\{\lambda_1 u_1(x_1) + \lambda_2 u_2(x_2) | x_1 \in \mathbb{R}^\ell_+, x_2 \in \mathbb{R}^\ell_+\right)$ and . $x_1 + x_2 = e_1 + e_2\}\}$. Define Pareto optimal allocations for this economy. Then prove that the set of all allocations $(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^{2\ell}_+$ solving the constrained maximization problem in the definition of $\hat{U}$ is the set of Pareto optimal allocations.

(f) Show an example of an economy where one point in the λ-simplex is sufficient to get the entire set of Pareto optimal allocations as the set of solutions of the maximization problem in the definition of $\hat{U}$. Can you find an example of an economy where you need the entire simplex, including the extreme points?

250 Rustichini

Question III.1 Fall 2017 majors

Consider a two-players game, played over infinitely many periods. In each period players are informed whether the current game is the Prisoner's Dilemma (PD) or the Battle of the Sexes (B oS). Then players choose simultaneously an action, and then the action profile thus chosen is publicly announced. Then the game for the next period is drawn, and is equal to the current game with probability $\alpha \in (0,1)$. After that, a new period begins with the announcement of the new game. Payoffs are discounted by $\delta \in (0,1)$. You have to speeify payoffs for the two games, consistent with the standard definition of PD and BoS; you are otherwise free to choose the utility functions.

(a) Formulate an extensive form game which is equivalent to this game; you do not need to have nature move at the initial node.

(b) Find at least four subgame perfect equilibria of the game. If you impose conditions on δ, state them explicitly and clearly.

(c) Suppose that at the moment of choosing the action each player has a memory constraint: the player can only remember the current game, the action profile played in the past period, and the game of last period.

(a) How many strategies for the infinitely repeated games does each player have, if the player has to satisfy the memory constraint?

(b) Find at least three subgame perfect equilibria of the game where the strategies of players are subject to the memory constraint. Specify the values of δ for which the strategy profiles are equilibria.

251 Rustichini

Question III.2 Fall 2017 majors

Let G be a normal form game with n players.

(a) Define a game with feedback (G, f). You have to state precisely what f is. Be very careful when you define the set of players.

(b) Define a self-confirming equilibrium for (G, f)

(c) Give an example of a game with feedback (G, f) with a self-confirming equilibrium such that the strategy profile of the equilibrium is not a Nash equilibrium of the game G.

(d) State and prove what is the relation between strategy profiles of self-confirming equilibria of (G, f) and the set of Nash equilibria of G if the feedback function includes information on the frequency of action profiles chosen by all players.

(e) State and prove what is the relation between strategy profiles of self-confirming equilibria of (G, f) and the set of Nash equilibria of G if the feedback function includes information on the frequency of utility profiles of all players. Be careful in stating the conditions on G.

(f) Consider now a game of incomplete information: players have payoff uncertainty, because the utility from the action profile depends on the realization of a type variable, θ^i for all i, with $\theta = (\theta_i)_{i \in I}$.θ^i is only communicated to i. What is the definition of the self-confirming equilibrium at a realization $\hat{\theta}$ of the vector of types that naturally extends the definition for the complete information case?

252 Rahman

Question IV.1 Fall 2017 majors

Vickrey Payments and the Core

(a) Consider an economy consisting of three identical buyers, a, b and c, and a seller, s, with three identical objects. Each buyer's utility function is given by $u(1) = 7, u(2) = 8$ and $u(3) = 10$. The seller's opportunity cost for the objects is zero.

(a) Compute the maximum gains from trade, $v(T)$, for every coalition $T \subseteq \{a, b, c, s\}$

(b) Find the core of this economy.

(c) An imputation of value to the individuals in this economy is called vector of Vickrey payments if each buyer's utility equals her marginal product and the seller's utility is the difference between the maximum gains from trade in the economy and the sum of the buyers' marginal products. Are the Vickrey payments in the core? Diseuss.

(b) Consider an economy consisting of two buyers, a and b, and a seller, s with four identical objects. Valuations are given by - Buyer a : $u_a(1) = 4, u_a(2) - 7, u_a(3) - 9, u_a(4) - 9$ - Buyer b : $u_b(1) = 4, u_b(2) = 7, u_b(3) = 9, u_b(4) = 10$ Seller s : $u_s(1) = u_s(2) = u_s(3) - u_s(4) = 0$ (a) Compute the maximum gains from trade, $v(T)$, for every coalition $T \subseteq \{a, b, c, s\}$

(b) Find the core of this economy.

(c) Are the Vickrey payments in the core?

(d) Is it possible to deliver these Vickrey payments with an anonymous market for objects? For bundles of objects?

253 Rahman

Question IV.2 Fall 2017 majors

Dynamic Mechanism Design

Consider and economy with one buyer and one seller of a single indivisible object. There are two periods of time. In the each period, the buyer learns independent pieces of information. In the first period, the buyer obtains a signal $\theta \in \{L, H\}$ such that $\Pr(\theta = H) = p$ and $\Pr(\theta = L) = 1 - p$ in the second period, the buyer obtains a signal $\sigma \sim U[0,1]$. The buyer's valuation for the good, $v_\theta(\sigma) \geq 0$, satisfies

$$v_L(\sigma) < v_H(\sigma) \quad \forall \sigma \in (0,1)$$

Assume that $v_\theta(\sigma)$ has a bounded, strictly positive derivative at every $\sigma \in (0,1)$, and that, for every $\sigma \in [0,1]$ as well as every $\theta \neq \theta'$, there is a unique $\sigma' \in [0,1]$ such that $v_\theta(\sigma) = v_{\theta'}(\sigma')$. The buyer's outside option and the seller's opportunity cost for the object are both equal to zero. A (direct) mechanism is a pair $(\ x, t \ )$ such that $x_\theta(\sigma) \in [0,1]$ for every (θ, σ) is the probability that the buyer receives the object as a function of his two pieces of information, and $t_\theta(\sigma) \in \mathbb{R}$ is the money paid by the buyer to the seller. Both parties have quasi-linear utility.

(a) Assume that σ is also observed by the seller, but not θ.

(a) Write down the buyer's incentive compatibility and individual rationality constraints.

(b) Assume that (x^*, t^*) maximizes the seller's expected revenue. Show that both the incentive constraint for $\theta = H$ pretending to be L and the individual rationality constraint for $\theta = L$ will bind. Show that, in addition,

$$\int_0^1 [v_H(\sigma) - v_L(\sigma)]\, [x_H^*(\sigma) - x_L^*(\sigma)]\, d\sigma \geq 0$$

(c) Use the binding constraints to show that the seller's problem simplifies to

$$\max_x \int_0^1 [p v_H(\sigma) x_H(\sigma) + (v_L(\sigma) - p v_H(\sigma))\, x_L(\sigma)]\, d\sigma \text{ s.t. } (1)$$

where the expected transfers are pinned down by the binding constraints.

(d) Find an optimal mechanism for the seller.

(b) Assume now that σ is not observed by the seller; recall the x^* you derived previously.

(a) Show that for there to exist transfers that make x^* incentive compatible in this new informational regime, $x_\theta^*(\sigma)$ must be increasing in σ for each θ

(b) Suppose that $v_L(\sigma)/v_H(\sigma)$ is strictly increasing in σ. Is x^* implementable?

(c) What if x^* is strictly decreasing?

254 Werner

Question I.1 Spring 2018 majors

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}_+^n$ in an economy with two agents:

$$\max_x \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$
$$\text{subject to } x \leq \omega, x \geq 0$$

where $u_i : \mathbb{R}_+^n \to \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for $i = 1, 2$. Let $x^* (\mu_1, \mu_2)$ be the set of solutions. (a) State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega - x)$ is supermodular in x

(b) Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^* (\mu_1, \mu_2)$ is non-decreasing in μ_1. You may assume that $x^*(\mu)$ is single-valued. Is $x^* (\mu_1, \mu_2)$ non-increasing in μ_2? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

(c) Under what conditions on u_1 and u_2 is the solution $x^* (\mu_1, \mu_2)$ unique. Justify your answer.

255 Werner

Question I.2 Spring 2018 majors

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values. You may assume that $\tilde{y}$ and $\tilde{z}$ take values in a finite interval $[a, b]$

(a) State a definition of $\tilde{z}$ second-order stochastically dominating (SSD) $\tilde{y}$. Your definition should be stated in terms of cumulative distribution functions F_y and F_z

(b) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Provide a justification for why it is a sensible definition of more risky.

(c) Extend the definition of more risky to random variables that may have different expected values as follows: $\tilde{y}$ is more risky than $\tilde{z}$ if and only if $\tilde{y} - E(\tilde{y})$ is more risky than $\tilde{z} - E(\tilde{z})$. Show that if $\tilde{y}$ is more risky than $\tilde{z}$ and $E(\tilde{z}) \geq E(\tilde{y})$, then $\tilde{z}$ second-order stochastically dominates $\tilde{y}$

(d) Show that $\lambda\tilde{z}$ is more risky than $\tilde{z}$ for every $\lambda \geq 1$ and every $\tilde{z}$. Note that $E(\tilde{z})$ may be different from 0

256 Allen

Question II.1 Spring 2018 majors

Consider a pure exchange economy with two agents, $i \in \{1, 2\}$ and two goods $m \in \{1, 2\}$. There exists a total of four units of each good in the economy. Each agent has identical preferences represented by the utility function $u_i (c_{i,1}, c_{i,2}) = \sqrt{c_{i,1}} + \sqrt{c_{i,2}}$. Both agents have an unbounded ability to eat any non-negative amount of either good.

(a) Sketch the utility possibilities frontier for this economy.

(b) Set up the "Social Planner's Problem" for this economy for characterizing the set of Pareto efficient allocations.

(c) Are all Pareto efficient allocations solutions to the Social Planner's problem? Are all solutions to the Social Planner's problem Pareto efficient? For both questions, if so, prove why so. If not, argue why not.

257 Allen

Question II.2 Spring 2018 majors

Consider a 2 good, 2 agent world. Good one, x, denotes oranges and good two, y denotes orange juice. Agent $i = 1$ has utility function $u_1(x, y)$ and agent $i = 2$ has utility function $u_2(x, y)$. Suppose each agent is endowed with 1 orange and no orange juice and an equal ownership share of each of two firms. Firm 1 can turn oranges into orange juice according to the production function $f_1(x) = \sqrt{x}$. Firm 2 can turn oranges into orange juice according to the production function $f_2(x) = 2\sqrt{x}$

(a) Define a Competitive Equilibrium.

(b) Assuming u_1 and u_2 satisfy the usual properties (strictly increasing in both arguments, concave, differentiable), derive a set of necessary equations for equilibrium. (Do not try to solve the system).

(c) Discuss which objects will be determined in equilibrium and which won't.

258 Rustichini

Question III.1 Spring 2018 majors

Let the set of consequences be $C = [0, 1]$. You have to extend the von Neumann Morgenstern representation theorem to this case. Specifically:

(a) Define a preference order $\succeq$ over lotteries in this case; distinguish between simple lotteries, which have a finite support, and lotteries.

(b) Assume that the restriction of $\succeq$ is monotonic in the order over the real numbers, that is $c \succeq d$ if and only if $c \geq d$

(c) Define the Continuity (C), Monotonicity (M), Reduction of Compound Lotteries (RCL) and Independence (I) axioms.

(d) Can you define a utility over consequences using M and C?

(e) Can you prove the von Neumann Morgenstern representation theorem in this case? If you claim you can, give a proof.

259 Rustichini

Question III.2 Spring 2018 majors

Let Γ be an extensive form game, x a node in Γ, such that the subgame beginning at $x, \Gamma(x)$, is a well defined extensive form game. Let η be a perturbation vector of the set of behavioral strategies in Γ.

(a) Provide a precise definition of what η is, what the induced perturbed extensive form game $\Gamma(\eta)$ is, and what $\Gamma(x, \eta)$ is.

(b) Let $\hat{\sigma}$ be a Nash equilibrium of the game $\Gamma(\eta)$, and consider the restriction of $\hat{\sigma}$ to $\Gamma(x, \eta)$. Define precisely what this restriction is, and prove that it is a Nash equilibrium of $\Gamma(x, \eta)$

260 Rahman

Question IV.1 Spring 2018 majors

Nonlinear Pricing

A firm produces a single good at constant marginal cost $c > 0$. The firm offers a pricing schedule $T(q)$ depending on the quantity $q \geq 0$ demanded by a consumer. The consumer has utility $\theta V(q) - T(q)$, where $V(0) = 0$ $V' > 0$ and $V'' < 0$, and $\theta \in \{\underline{\theta}, \bar{\theta}\} \subset (0, \infty)$ with $\underline{\theta} < \bar{\theta}$ (a) Define and solve the firm's problem assuming that θ is common knowledge.

From now on, assume that the consumer knows θ but the firm does not; it believes that $\underline{\theta}$ occurs with probability $\underline{p} \in (0, 1)$ and $\bar{\theta}$ with probability $\bar{p} = 1 - \underline{p}$. The consumer can always consume zero and walk away, earning a utility of zero.

(b) Write down the consumer's individual rationality and incentive compatibility constraints depending on his type. Write down the firm's profit-maximization problem.

(c) Show that, at a profit-maximizing pricing schedule, only the low type's individual rationality constraint and the high type's incentive compatibility constraint bind.

(d) Use the observations in (c) to solve the firm's profit maximization problem assuming that $V(q) = \ln(1 + q)$ and $\bar{p}\bar{\theta} < \underline{\theta}$. What happens if $\bar{p}\bar{\theta} \geq \underline{\theta}$?

261 Rahman

Question IV.2 Spring 2018 majors

Dynamic Moral Hazard

Consider a two-period principal-agent problem. In each period, there are m possible output levels for the principal, $\pi_i \in \mathbb{R}$, and n effort levels for the agent, $e_j \in \mathbb{R}$. Let $\Pr(\pi|e)$ be the probability of π given e. The principal's per-period payoff is expected output minus payments to the agent. The agent's is

$$U(z, a) = v(z) - c(a)$$

where $v' > 0, v'' < 0, c' > 0, c'' > 0, v(z)$ corresponds to the agent's utility from payments by the principal and $c(a)$ corresponds to effort costs. Time elapses as follows. First, the principal makes a take-it-or-leave-it offer to the agent, who has an outside option worth U. Once everyone agrees to the contract, there are no more opportunities to quit throughout the two-period relationship. In period 1, the agent exerts effort e_1, output π_1 realizes, and the agent is paid $w_1(\pi_1)$. Everyone observes π_1 at the end of period 1. In period 2, the agent exerts effort e_2, output π_2 realizes, and the agent is paid $w_2(\pi_1, \pi_2)$ Everyone's overall payoff is the sum of per-period payoffs.

(a) Write down formally the principal's problem of (i) minimizing the cost of implementing a given effort profile $\mathbf{e} = (e_1, \tilde{e}_2)$, where $e_1 \in \mathbb{R}$ is period- 1 effort and $\bar{e}_2 \in \mathbb{R}^m$ is the effort plan in period 2 contingent on output in period 1 subject to a lifetime participation constraint, and (ii) maximizing expected profit.

(b) Show that, in general, w_2 will depend not only on π_2, but also on π_1.

(c) Derive and interpret the inverse Euler equation from (a-i):

$$\frac{1}{v'(w_1(\pi_1))} = \sum_{\pi_2} \frac{1}{v'(w_2(\pi_1, \pi_2))} \Pr(\pi_2 | e_2(\pi_1))$$

262 Werner

Question I.1 Fall 2018 majors

State a definition of the lexicographic preference relation on the consumption set R_+^2 with the first priority for total consumption of goods 1 and 2, and the second priority for good 2

(a) Show that this preference order is not continuous. Clearly state a definition of continuity.

(b) Does this preference relation have a utility representation on R_+^2? Justify your answer.

(c) Derive Walrasian demand function $x^*(p_1, p_2, w)$ of prices $p_1 > 0, p_2 > 0$ and income $w > 0$ for this lexicographic preference.

(d) State the Weak Axiom of Revealed Preference. Does the demand function you derived in (c) satisfy the weak axiom? Justify your answer.

263 Werner

Question I.2 Fall 2018 majors

Consider the utility function U on state-contingent consumption plans on S states with probabilities $\{\pi_n\}$ defined by

$$U(c) = E(c) - \alpha \operatorname{var}(c)$$

for $c \in R^S$, where $E(c)$ and var(c) denote the expected value and the variance of e, respectively, and $\alpha > 0$ is a parameter. (a) Derive risk compensation for this utility function for any risky claim $\tilde{z} \in R^S$ with $E(\tilde{z}) = 0$ Suppose that there are two assets: a risk-free asset with return r_f and a risky asset with return $\tilde{r}$. Returns are per-dollar returns, that is, total returns for one dollar invested. The agent has utility function U and initial wealth $w > 0$. Negative investment (i.e., short selling) is permitted for both assets.

(b) Find the optimal investment in the risky asset as a function of the expected return, the variance of the return on the risky asset, the risk-free return, and the agent's wealth. Consider the following comparative statics exercise. The return $\tilde{r}$ is changed to a more risky return $\tilde{r}'$. The expected return on $\tilde{r}'$ is the same as on $\tilde{r}$, that is, $E(\tilde{r}') = E(\tilde{r})$

(c) State a definition of more risky return (or more risky random variable).

(d) Suppose that $E(\tilde{r}) > r_f$. Show that the optimal investment in the risky asset with more risky return $\tilde{r}'$ is smaller then the optimal investment with less risky return $\tilde{r}$, everything else being unchanged? Prove your answer.

264 Phelan

Question II.1 Fall 2018 majors

Suppose nature is going to flip a coin. With probability $\pi \in (0,1)$ the coin is heads (H), and agent 1 receives an endowment of 2 bananas and agent 2 receives an endowment of 1 banana, and with probability 1 - π the coin is tails (T), and agent 1 receives an endowment of 0 bananas and agent 2 receives an endowment of 1 banana. There is no production and all endowments are observable. Let $s \in \{H, T\}$ be the joint endowment realization, thus $e_1(s) = 2$ if $s = H, e_1(s) = 0$ if $s = T$, and $e_2(s) = 1$ for $s \in \{H, T\}$. Assume preferences for agent 1 are characterized by $\pi\sqrt{c_1(H)} + (1-\pi)\sqrt{c_1(T)}$. Assume preferences for agent 2 are characterized by $\pi c_2(H) + (1-\pi)c_2(T)$. Assume agent 1 can eat any non-negative amount and agent 2 can eat any amount.

(a) What is an allocation in this environment? What is a feasible allocation?

(b) Characterize the set of Pareto efficient allocations. Put in words what is necessary and sufficient for efficiency.

(c) Characterize the competitive equilibrium from these endowments.

265 Phelan

Question II.2 Fall 2018 majors

Consider a pure exchange economy with two agents, $i \in \{1, 2\}$ and two goods $m \in \{1, 2\}$. There exists a total of four units of each good in the economy. Agent 1's preferences are represented by $u_1(c_{1,1}, c_{1,2}) = \sqrt{c_{i,1}} + \sqrt{c_{1,2}}$ and agent 2 's preferences are represented by $u_2(c_{2,1}, c_{2,2}) = \sqrt{c_{2,1}}$. Both agents have an unbounded ability to eat any non-negative amount of either good.

(a) Sketch the utility possibilities frontier for this economy.

(b) Set up the "Social Planner's Problem" for this economy for characterizing the set of Pareto efficient allocations.

(c) Are all Pareto efficient allocations solutions to the Social Planner's problem? Are all solutions to the Social Planner's problem Pareto efficient? For both questions, if so, prove why so. If not, argue why not.

266 Rustichini

Question III.1 Fall 2018 majors

(a) State and prove Zermelo's theorem.

(b) Illustrate the theorem in a simple example.

(c) Discuss whether uniqueness holds under the conditions stated for the theorem; provide either proof or counterexample. Can you find sufficient condition for uniqueness?

(d) What does the theorem imply for the games of checkers and chess? Prove your statement.

(e) Can you make the implications of the theorem sharper for those two games?

(f) Consider whether Zermelo's theorem can be generalized. In particular, note that the theorem has two assumptions: can you relax any of the two and still prove that the conclusion holds? For each, provide either a proof of your statement or a counterexample.

(g) For at least one of the two assumptions, can you find additional conditions that are sufficient for the conclusion to hold?

267 Rustichini

Question III.2 Fall 2018 majors

Consider the following extensive form game with nature and two players. Nature moves first, and draws H with probability p or L; this is called the state. This outcome is communicated to the first player, and only to him. He can then choose In or Out. If he chooses Out the game is over and the two players collect their payoff. If he chooses In, the move goes to the second player, who can then choose h or l (that is, he is asked to guess the initial choice of nature.) After that the game is over and payoff depend on the true choice of nature and the second player's guess. This completely deseribes the extensive form game, except for the payoffs, that are discussed in the following.

(a) Draw the extensive form game. Define pure, mixed and behavioral strategies for the game;

(b) Characterize the equilibrium set for the zero-sum game game where i The payoff in the final nodes following the Out decision are the same independent of the state; ii The payoffs after the In decision only depend on whether the second player guesses correctly (l is chosen in the state L, h in H), or wrongly;

(c) Characterize the equilibrium set in the general, not necessarily zero-sum, case where the restrictions (i) and (ii) in point b above hold

(d) Define sequential equilibria for extensive from games.

(e) Can you find an example of a game (that is, of final payoffs) with multiple equilibria? Can you find an example of a game with an equilibrium which is not a sequential equilibrium?

268 Rahman

Question IV.1 Fall 2018 majors

Private versus Common Values in a First Price Auction Consider a first-price sealed-bid auction between two bidders for a single indivisible object. Before the auction, each bidder $i \in \{1,2\}$ observes an IID random variable $\theta_i \sim U[0,1]$. The value of the object to bidder i is given by $v_i = \theta_i + \frac{1}{2}$. (This is known as independent private values.) After observing θ_i, each bidder i submits a sealed bid $b_i \in [0, \infty)$. Bidder i^is payoff if bidders observe $\theta = (\theta_1, \theta_2)$ and bid $b = (b_1, b_2)$ is given by

$$U_i(b|\theta) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

where $j \neq i$. Intuitively, the object is awarded to the highest bidder, who pays what he or she bid. In case of a tie, that is, both bidders made the same bid, the object is awarded according to the toss of a fair coin. (a) Assume that bidder j 's bidding strategy is given by $b_j = \alpha\theta_j + \beta$, for some constants α and β, and that $j \neq i$ below.
What is the expected payoff to bidder i who observed θ_i from bidding b_i conditional on winning the object? ii What is the probability that a bid b_i by bidder i wins the object? iii What bid b_i maximizes bidder i' s expected payoff given θ_i?

(b) Find an equilibrium in which each bidder i uses a bidding strategy of the form $b_i = \alpha\theta_i + \beta$. In this equilibrium, what is the expected payoff to bidder i given his type θ_i?

(c) Repeat all of the above assuming independent common values, that is, each bidder i 's value of the object is instead given by $v_i = \theta_i + \theta_j$, where $j \neq i$. Thus, given i 's own information θ_i, the expected value of the object to bidder i is still $\theta_i + \frac{1}{2}$

(d) Are equilibrium bids lower or higher under private versus common values?

269 Rahman

Question IV.2 Fall 2018 majors

The Core and the Shapley Value Consider a simplification of the United Nations (UN) consisting of five member countries: $N = \{1, 2, 3, 4, 5\}$. Countries 1 and 2 are big, whereas countries $3, 4$ and 5 are small. The UN can approve an action on any issue relating to international security provided both big countries and at least one small country vote for the proposed action.

(a) Describe the power structure of this version of the UN as a simple game, that is, a cooperative game with values in $\{0, 1\}$, where a coalition has value 1 if and only if a vote for the proposed action by every member of the coalition approves the action.

(b) What is the core of this game?

(c) What is the Shapley value of this game? Recall that the Shapley value φ of a cooperative game $v : 2^N \to \mathbb{R}$ is given by

$$\varphi_i(v) = \sum_{S \in N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S))$$

for every $i \in N$. Hint: In this simple game, $v(S)$ is either 0 or 1, and $v(S \cup \{i\}) - v(S)$ is either 0 or 1. Moreover, $v(S \cup \{i\}) - v(S) = 1$ only if player i is pivotal in S, that is, $v(S \cup \{i\}) = 1$ but $v(S) = 0$. Restricted to simple games, the Shapley value is also known as the Shapley-Shubik power index.

270 Werner

Question I.1 Spring 2019 majors

Consider a demand function $d(p, w)$ of prices $p \in \mathbb{R}^L$ and income w where $p \gg 0$ and $w \geq 0$, satisfying budget identity (or Walras law) $pd(p, w) = w$

(i) State the Law of Compensated Demand for d.

(ii) Suppose that d is a Walrasian demand function obtained from maximizing some utility function subject to the budget constraint. Under what conditions on the utility function does d satisfy the Law of Compensated Demand? Prove your statement.

(iii) Consider the following demand function for two goods: Demand $d\,(p_1, p_2, w) -$ for $p_1 > 0, p_2 > 0$, and $w \geq 0$ - equals $\left(\frac{w}{p_1}, 0\right)$ if $p_1 \geq p_2$ and $\left(0, \frac{w}{p_2}\right)$ if $p_2 > p_1$ Does the Law of Compensated demand hold for d? Justify your answer. (iv) Can the demand function d from part (iii) be a Walrasian demand for some utility function? Justify your answer.

271 Werner

Question I.2 Spring 2019 majors

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\bar{z})$ and $E(\bar{y})$ their expected values. You may assume that $\bar{y}$ and $\bar{z}$ take values in a finite interval $[a, b]$

(i) State a definition of $\bar{z}$ first-order stochastically dominating (FSD) $\tilde{y}$. State a definition of second-order stochastic dominance (SSD). Your definitions should be stated in terms of cumulative distribution functions F_y and F_z

(ii) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.

(iii) Suppose that $\bar{z}$ has uniform distribution on an interval $[\underline{z}, \bar{z}]$ while $\bar{y}$ has uniform distribution on $[y, \bar{y}]$. Under what conditions on the bounds $\underline{z}, \bar{z}, \underline{y}, \bar{y}$ does $\bar{z}$ FSD $\tilde{y}$? Prove your statement. (iv) Suppose again that $\bar{z}$ and $\tilde{y}$ have uniform distributions as in (iii). Show that if $\underline{y} \leq \underline{z}$ and $z \leq \bar{y}$ and $E(\bar{z}) = E(\bar{y})$, then $\tilde{y}$ is more risky than $\bar{z}$ You may assume some specific (distinet) numerical values of $\underline{z}, \bar{z}, \underline{y}, \bar{y}$ in your proof, if you find it convenient.

272 Phelan

Question II.1 Spring 2019 majors

Consider a pure exchange economy with two agents, $i \in \{1, 2\}$ and two goods $m \in \{1, 2\}$. There exists a total of 2 units of each good in the economy. Each agent has identical preferences represented by the utility function $u_i\left(c_{i,1}, c_{i,2}\right) = c_{i,1}^2 + c_{i,2}^2$. Both agents have an unbounded ability to eat any non-negative amount of either good.

(a) Sketch the utility possibilities frontier for this economy.

(b) Set up the "Social Planner's" (or Negishi) Problem for this economy for characterizing the set of Pareto efficient allocations.

(c) Are all Pareto efficient allocations solutions to the Social Planner's problem? Are all solutions to the Social Planner's problem Pareto efficient? Explain.

(d) Suppose each agent is endowed with 1 unit of each good. Define a Competitive Equilibrium and give the set of Competitive Equilibria for this economy. Are they all Pareto Efficient? If not, what assumption of the 1st Welfare Theorem is violated?

(e) Can all Pareto Efficient Allocations in this environment be supported as Competitive Equilibria? If not, what assumption of the 2 nd Welfare Theorem is violated?

273 Phelan

Question II.2 Spring 2019 majors

Consider an economy populated by a unit continuum of agents with names $i \in [0,1]$. (The fraction of agents with names between x and y is equal to $|x-y|$). Each agent's endowment consists of 1 apple and a variable number of potatoes. Specifically, with i.i.d. probability $\frac{1}{2}$, an agent is endowed with 0 potatoes, and with probability $\frac{1}{2}$, the agent is endowed with 2 potatoes. Assume that exactly half the agents get an endowment of 0 potatoes and half get an endowment of 2 potatoes. (What is variable is which agents get the higher endowment.) Each agent has identical preferences and cares about the expected value of $U_i\left(c_{i,A}, c_{i,P}\right) = u\left(c_{i,A}\right) + u\left(c_{i,P}\right)$ where u is strictly concave and where $c_{i,A}$ and $c_{i,P}$ are his consumption levels of apples and potatoes respectively. All agents have an unbounded ability to eat any non-negative amount of either good.

(a) Define a symmetric allocation as one where an agent's consumption may not depend on his name. When is such an allocation feasible? Characterize the set of symmetric Pareto efficient allocations.

(b) Redefine symmetric allocations in terms of transfers. When is such an allocation feasible? Characterize the set of symmetric Pareto efficient allocations.

(c) Now suppose that because potatoes grow underground, that an agent with an endowment of 2 potatoes can falsely claim he has an endowment of 0 potatoes and eat his hidden potatoes when no one is looking. Define an incentive compatible symmetrical allocation. Characterize the set of incentive compatible symmetric Pareto efficient allocations.

(d) Finally, allow an allocation to depend on an agent's name. When is such an allocation feasible? Characterize the set of Pareto efficient allocations under both when potatoes are observable, and when they can be hidden.

274 Rustichini

Question III.1 Spring 2019 majors

A normal form game is called symmetric if each player has the same set of pure strategies and for every player i and any permutation π of the players index

$$u^i\left(a_1, \ldots, a_n\right) = u^{\pi(i)}\left(a_{\pi(1)}, \ldots, a_{\pi(n)}\right)$$

(a) Write an example of a 3 players and an example of a 4 players symmetric game where all the entries for a player are different.

(b) Show that any symmetric game has a symmetric Nash equilibrium, that is a Nash equilibrium where every player chooses the same strategy.

(c) Find the symmetric equilibrium of your example with 3 players.

(d) Are all the equilibria of a symmetric game symmetric? Prove or disprove your claim.

(e) Prove that the value of a symmetric two-players, zero-sum game is zero.

(f) In a two players symmetric game, let A denote the payoff matrix of the first player, and denote x, y the mixed strategies. A mixed strategy x is an evolutionarily stable strategy if and only if for some ϵ^* and all $\epsilon < \epsilon^*$ and all mixed strategies $y \neq x$

$$x^T A((1-\epsilon)x + \epsilon y) > y^T A((1-\epsilon)x + \epsilon y)$$

Do evolutionarily stable strategy always exist? prove or disprove your statement.

275 Rustichini

Question III.2 Spring 2019 majors

(a) Consider the game

	a	b	c	d
A	$7,3$	$6,3$	$5,5$	$4,7$
B	$4,2$	$5,8$	$8,6$	$5,8$
C	$6,1$	$3,8$	$2,4$	$6,9$

Find all the sets that survive iterated elimination of weakly dominated strategies, and of strictly dominated strategies. ii Find the set of Nash equilibria.

(b) Consider the game

	a	b
A	$1,1$	$5,3$
B	$3,0$	$5,3$
C	$1,1$	$0,4$
D	$3,0$	$5,3$

Is this game the normal form game of some extensive form game? If you state it is, then describe the extensive form game. If you state it is not, then provide a proof of your statement.

(c) For a general extensive form game: Present an example of an extensive form game that does not have an equilibrium in pure strategies, but where each player has at least one information set which is a singleton.

ii In a finite extensive form game, let Y be a non-empty subset of the set of nodes of the tree X, with the partial order induced by the restriction of the order of the tree. Prove or disprove: there is an element in Y with no immediate successors.

iii Prove that in a finite extensive form game of perfect information there is a node x which has all the immediate successors in the set of final nodes.

iv Prove that in an extensive form game of perfect information every node defines a subgame.

276 Rahman

Question IV.1 Spring 2019 majors

The City of Minneapolis is considering whether to erect a replica of Michelangelo's statue of David in the middle of Loring Park. Some residents argue that it is high time we had classic art competing for attention with giant spoons and blue chickens. Others are uncomfortable with nudity, even if artistic, in the public sphere. Let $X = \{0,1\}$ be the set of alternatives (whether or not to build the statue). There are $n \in \mathbb{N}$ individuals. Each individual i 's utility over X is $v_i(t_i, 0) = 0$ and $v_i(t_i, 1) = t_i$, where $t_i \in \mathbb{R}$ is a real number.

(a) Describe all Groves schemes for this problem.

(b) Write down Clarke's scheme. Is there a budget deficit? How much? When?

(c) For any function $f : \mathbb{R}^n \to \mathbb{R}$ and $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, define

$$\Delta_i^{si} f(t) = f(t) - f(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)$$

Show that a Groves scheme satisfies ex post budget balance if and only if for every pair of type profiles $t = (t_1, \ldots, t_n)$ and $s = (s_1, \ldots, s_n)$

$$\Delta_n^{s_n} \cdots \Delta_2^{2_2} \Delta_1^{s_2} W(t) = 0$$

$(*)$ where, $W(t) = \sum_i v_i(t_i, x^*(t))$ and x^* is an ex post efficient allocation. Find a pair of type profiles t and s that violate Condition $(*)$

277 Rahman

Question IV.2 Spring 2019 majors

Consider an economy with two men, m' and m'', and two women, w' and w''. When a man and a woman join forces, they generate surplus according to the table below:

	$w\prime$	w'
$m\prime$	4	7
m''	7	9

Only men and women can match together in this economy to generate any surplus, in unit amounts.

(a) What is the surplus-maximizing assignment of men to women?

(b) Assuming that utility is transferable (i.e., individuals have quasilinear preferences with respect to some money commodity), describe the core of this assignment problem. What are the possible core imputations of value to man m'? To man m''? To woman w'? To woman w''?

(c) Now assume that utility is not transferable. Let

$$\begin{aligned} U\left(m', w'\right) &= \left\{\pi \in \mathbb{R}_{+}^{2} : \pi_{m'} \leq 2, \pi_{w'} \leq 2\right\} \\ U\left(m', w''\right) &= \left\{\pi \in \mathbb{R}_{+}^{2} : \pi_{m'} + \pi_{w''} \leq 7, \pi_{w''} \leq 3.5\right\} \\ U\left(m'', w'\right) &= \left\{\pi \in \mathbb{R}_{+}^{2} : \pi_{m''} + \pi_{w'} \leq 7, \pi_{m''} \leq 3.5\right\} \\ U\left(m'', w''\right) &= \left\{\pi \in \mathbb{R}_{+}^{2} : \pi_{m''} + \pi_{w''} \leq 9\right\} \end{aligned}$$

be the utility possibility sets enjoyable by any man and woman who match with one another. Consider the problem of assigning men to women and giving value to each individual feasibly, i.e., so that, whenever man m is matched with woman w, the value to each party, $\pi(m, w) = (\pi_m, \pi_w) \in \mathbb{R}_{+}^{2}$, belongs to the set $U(m, w)$. An assignment of men to women together with an imputation of value is called a stable outcome if no man m and woman w exist who are not matched to each other but who would do better (at least one individual would be strietly better off in terms of imputed value and nobody would be worse off) by abandoning their current match and joining forces with some new imputed value $\pi'(m, w) \in U(m, w)$

i. Show that the assignment of part (a) is not part of any stable outcome.

il. Find a stable outcome for this economy without transferable utility.

278 Werner

Question I.1 Fall 2019 majors

Consider a finite set of observations $\{p^t, y^t\}_{t=1}^T$ of pairs of price vectors and production plans of a firm, where $p^t \in R^L$ and $y^t \in R^L$. Suppose that $p^t \geq 0, p^t \neq 0$, and $p^t y^t \geq 0$ for every t. The Weak Axiom of Profit Maximization (WAPM) states that $p^t y^t \geq p^t y^s$ for every $s, t = 1, \ldots, T$
Production set Y is said to profit rationalize observations $\{p^t, y^t\}_{t=1}^T$ if $y^t \in Y$ and $p^t y^t = \max\{p^t y : y \in Y\}$ for every t

(i) Show that if observations $\{p^t, y^t\}_{t=1}^T$ satisfy WAPM, then the convex production set with free-disposal and zero-production

$$Y = \text{co}\left\{0, y^1, \ldots, y^T\right\} - R_+^L$$

rationalizes these observations. Here, "co" denotes the convex hull.

(ii) Show that production set Y exhibits non-increasing returns to scale.

(iii) Is y^t the unique profit-maximizing production plan in Y at prices p^t? Justify your answer. If your answer is "no," state a condition on observations $\{p^t, y^t\}_{t=1}^T$ under which y^t is the unique profit maximizer under p^t

(iv) Let π_Y be the profit function of the production set Y. Show that π_Y is a convex function. What is the set of price vectors for which π_Y is well defined (i.e., it takes finite values)?

279 Werner

Question I.2 Fall 2019 majors

(i) State the independence axiom (also called independence of common consequences for a preference relation on state-contingent claims (or acts) on a finite state space S

(ii) Show that a preference relation with expected utility representation satisfies the independence axiom.

(iii) Describe the Ellsberg paradox (with single urn containing balls of three different colors). Show that the typical pattern of preferences among bets in the Ellsberg paradox violates the independence axiom.

(iv) Suppose that an agent's preferences over bets are described (or represented) by multiple-prior expected utility with some strictly increasing von Neuman-Morgenstern (or Bernulli) utility function. Specify a set of probabilities of balls of different color being drawn from the urn such that the multiple-prior expected utility conforms to the typical pattern of preferences in the Ellsberg paradox. The set of probabilities should be such that balls whose fraction of the total number in the urn is unambiguously known have the probability equal to that fraction. All probabilities in the set should be strictly positive. You may assume that the vNM utility function is concave or strictly concave.

(v) Consider the following lexicographic preference relation over bets. The first priority is the expected utility of a bet under equal probabilities for balls of every color. Assume that the vNM utility function is strictly increasing and strictly concave. The second priority is the payoff of the bet when a ball of the (single) color whose fraction is unambiguously known is drawn. Does this lexicographic preference relation conform to the pattern of preferences in the Ellsberg paradox? Does it satisfy the independence axiom? Justify.

280 Phelan

Question II.1 Fall 2019 majors

Consider a pure exchange economy with two agents, $i \in \{1,2\}$ and two goods $m \in \{1,2\}$. There exists a total of 3 units of each good in the economy. Each agent has identical preferences represented by the utility function $u_i(c_{i,1}, c_{i,2}) = \min\{2, c_{i,1}\} + \min\{2, c_{i,2}\}$. Both agents have an unbounded ability to eat any non-negative amount of either good.

(a) Carefully characterize the set of Pareto Efficient allocations for this economy and sketch the utility possibilities set for this economy.

(b) Set up the "Social Planner's" (or Negishi) Problem for this economy.

(c) Are all Pareto Efficient allocations solutions to the Social Planner's problem? Are all solutions to the Social Planner's problem Pareto Efficient? Explain.

(d) Define a Competitive Equilibrium for this economy and give the set of Competitive Equilibria for all possible endowment specifications subject to the aggregate endowment being 3 for each good. Are they all Pareto Efficient? Are any Pareto Efficient. If not, why not?

(e) Can all Pareto Efficient Allocations in this environment be supported as a Competitive Equilibrium for some set endowments (again where the aggregate endowment is 3 for each good)? If not, what assumption of the 2nd Welfare Theorem is violated?

281 Phelan

Question II.2 Fall 2019 majors

Consider an economy populated by N agents, $i \in \{1, \ldots, N\}$. With i.i.d. probability $\frac{1}{2}$, each agent is endowed with 0 potatoes, and with probability $\frac{1}{2}$, the agent is endowed with 2 potatoes. Each agent has identical preferences and cares about the expected value of $U_i(c_i) = u(c_i)$ where u is strictly concave and where c_i is his consumption of potatoes. All agents have an unbounded ability to eat any non-negative amount of potatoes.

(a) Define an allocation. When is such an allocation feasible?

(b) Characterize the set of Pareto efficient allocations. In particular, prove that each agent's consumption is a function only of the aggregate endowment of potatoes.

(c) Assume $u(c) = c^{1-\sigma}/(1-\sigma)$. Prove that each agent's consumption is a constant share of the aggregate endowment of potatoes.

(d) Finally assume agent 1 's endowment of potatoes is private to him. Under what conditions (if any) is a Pareto Efficient allocation assuming full information (or full observability) incentive compatible?

282 Rustichini

Question III.1 Fall 2019 majors

(a) A normal form game is called symmetric if each player has the same set of pure strategies and for every player i and any permutation π of the players index

$$u^i\left(a_1, \ldots, a_n\right) = u^{\pi(i)}\left(a_{\pi(1)}, \ldots, a_{\pi(n)}\right)$$

Write an example of a 3 players and an example of a 4 players symmetric game where all entries for each player are different.

(b) Are all the equilibria of a symmetric game symmetric? Prove or disprove your claim.

(c) Consider a symmetric two players game, with action set A (the same for both players). Denote $x \in \Delta(A)$ a mixed strategy, and interpret $x(a)$ as the frequency of players playing action a in a population. The utility to a player of playing the action a when the frequency over the actions is x is denoted $u(a, x) \equiv \sum_{b \in A} u(a, b)x(b)$, and the average payoff is $\bar{u}(x) \equiv \sum_{a \in A} u(a, x)$. Prove that a symmetric equilibrium (that is, a $\hat{x}$ such that $\hat{x} \in BR(u(\cdot, \hat{x}))$ exists.

(d) Replicator dynamics describes the evolution over time of the frequency of a strategy, depending on the payoffs obtained. So we now consider the frequency of action a at time t, denoted by $x_t(a)$, and we let $\frac{dx_t(a)}{dt}$ be the derivative with respect to time. The equation describing the evolution over time is

$$\forall_{a \in A} : \frac{dx_t(a)}{dt} = x_t(a)\left(u\left(a, x_t\right) - \bar{u}\left(x_t\right)\right) \equiv f\left(x_t\right)$$

A steady state of the equation is an x^* such that $f\left(x^*\right) = 0$

i Prove that a steady state always exists;
ii State what is the relationship between steady states and Nash equilibria of the game. Prove your answer, providing if necessary examples to illustrate possible differences between the two.

283 Rustichini

Question III.2 Fall 2019 majors

(a) Consider the game

	a	b	c	d
A	$1, 8$	$7, 9$	$1, 2$	$5, 8$
B	$15, 20$	$13, 1$	$21, 2$	$13, 1$
C	$5, 5$	$3, 5$	$7, 1$	$3, 4$

i Find all the sets that survive iterated elimination of weakly dominated strategies, and of strictly dominated strategies.
ii Find the set of Nash equilibria.

(b) i In the following normal form game, letters indicate pairs of payoff (so you can think of a say as $(3, 2)$. This notational choice is meant to make the question more transparent. Consider the game

	L	R
A	a	d
B	e	e
C	b	c
D	e	e

Find an extensive form game that has this game as its normal form, and has at most five final nodes.
ii Define what a linear extensive form game is, and give an example of a game which is not linear. iii Is the following statement true?

For every extensive form game G which is not linear there is a linear extensive form game that has the same normal form. You should prove your answer or provide a counterexample.

(c) Find a symmetric two players normal form game, where each player has three actions that has seven Nash equilibria, each with a different payoff vector.

(d) Find a symmetric two players normal form game, with action set $\{A, B, C\}$, where the game restricted to the action set $\{A, B\}$ has a Nash equilibrium which is not a Nash equilibrium of the complete game.

284 Rahman

Question IV.1 Fall 2019 majors

Consider the following three-player game. Player 1 has two actions, $A_1 = \{U, D\}$. Player 2 also has two actions, $A_2 = \{L, R\}$. Player 3 has infinitely many actions, $A_3 = \mathbb{Z}$, with typical element $z \in \mathbb{Z}$. Payoffs are depicted below.

	L	R
U	$2, 2, 3 + 1/z$	$0, 0, 8$
D	$0, 0, 0$	$2, 2, 0$

$z < 0$

	L	R
U	$2, 2, 2$	$0, 0, 0$
D	$0, 0, 0$	$2, 2, 2$

$z = 0$

	L	R
U	$2, 2, 0$	$0, 0, 0$
D	$0, 0, 8$	$2, 2, 3 - 1/z$

$z > 0$

Intuitively, player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. Thus, if players 1 and 2 choose (U, L) and player 3 chooses $z = -4$ then players 1 and 2 each get a payoff of 2 and player 3 gets 2.75

(a) (i) Characterize the set of pure-strategy Nash equilibria. Characterize the set of mixed-strategy Nash equilibria.

(b) (i) Find a correlated equilibrium.
(ii) Find an information structure such that the correlated equilibrium distribution in (b-i) agrees with a Bayes-Nash equilibrium distribution of behavior in the game augmented by the information structure.

285 Rahman

Question IV.2 Fall 2019 majors

Let N be a finite set of players. A simple game is a family $\mathbb{D}$ of so-called winning subsets of N. For example, simple majority with n voters is a simple game with winning subsets $\{D : |D| > n/2\}$. A game is gridlocked if there is a subset of N such that neither it nor its complement is winning. Thus, simple majority is gridlocked if and only if n is even. Individual i is pivotal if it belongs to every winning subset, that is, $i \in D$ for all $D \in \mathbb{D}$. A game is free if it has no pivotal individuals, and dictatorial if it has only one. For example, the game $\mathbb{D}_P = \{S \subset N : P \subset S\}$ is free if and only if P is empty, and dictatorial if and only if $|P| = 1$
(a) Prove that if a simple game is neither free nor gridlocked then it is dictatorial.

286 Rahman

Question IV.2 Fall 2019 majors

Given a simple game $\mathbb{D}$, define its Nakamura number as

$$\nu(\mathbb{D}) = \min_{\mathcal{F} \subset \mathbb{D}} \{|\mathcal{F}| : \bigcap \{F : F \in \mathbb{F}\} = \emptyset\}$$

with $\nu(\mathbb{D}) = \infty$ if the game is not free. The number ν computes the smallest number of winning subsets with empty intersection. For example, the Nakamura number of simple majority with n voters is 3 if $n = 3$ or $n > 4$. A simple game $\mathbb{D}$ induces a social welfare function by the definition aPb if $\{i \in N : aP_ib\} \in \mathbb{D}$. Let A be a finite set of alternatives. Prove that P is acyclic whenever P_i is acyclic for every individual i if and only if $\nu(\mathbb{D}) > |A|$

287 Werner

Question I Fall 2020

Let $\succeq$ be a preference relation on R_+^L for arbitrary $L \geq 1$

(a) State a theorem providing sufficient conditions for $\succeq$ to have a utility representation. Clearly state all assumptions of the theorem.

(b) Prove your theorem under an additional assumption that $\succeq$ is strictly increasing (i.e., strongly monotone).

(c) For each property of the preference relation assumed in the theorem you proved in part (b) - with exception of the monotonicity assumption - show that it is essential. That is, demonstrate that the theorem does not hold if the assumption is dropped. For the assumption of continuity, and if you provide an example, you only need to sketch the argument.

288 Phelan

Question II Fall 2020

Consider a pure exchange economy with two agents, $i \in \{1, 2\}$ and two goods $m \in \{1, 2\}$. There exists a total of 25 units of each good in the economy.
Agent 1 has preferences represented by the utility function $u_1(c_{1,1}, c_{1,2}) = \sqrt{c_{1,1}} + \sqrt{c_{1,2}}$.
Agent 2 has preferences represented by the utility function $u_2(c_{2,1}, c_{2,2}) = \sqrt{\min(16, c_{2,1})} + \sqrt{\min(16, c_{2,2})}$.
Both agents have an unbounded ability to eat any non-negative amount of either good.

(a) Carefully characterize the set of Pareto Efficient allocations for this economy and sketch the utility possibilities frontier for this economy.

(b) Set up the "Social Planner's" (or Negishi) Problem for this economy.

(c) Are all Pareto Efficient allocations solutions to the Social Planner's problem? Are all solutions to the Social Planner's problem Pareto Efficient? Explain.

(d) Define a Competitive Equilibrium and give the set of Competitive Equilibria for this economy for all possible endowment specifications subject to the aggregate endowment being 25 for each good. Are they all Pareto Efficient? Are any Pareto Efficient. If not, why not?

(e) Can all Pareto Efficient Allocations in this environment be supported as a Competitive Equilibrium for some set endowments (again where the aggregate endowment is 25 for each good)? If not, what assumption of the 2nd Welfare Theorem is violated?

289 Rustichini

Question III Fall 2020

(a) Find the mixed strategy equilibria of the following game:

	L	R
T	1, 4	4, 3
M	2, 0	1, 2
B	1, 5	0, 6

(b) Consider the following game:

	C	D
A	x, y	3, 0
B	6, 2	0, 4

i Set $x = y = 2$ and find the Nash equilibria.

ii Find values of (x, y), if any, such that (α^1, α^2) is a mixed strategy Nash, where $\alpha^1(A) = 1/5, \alpha^2(C) = 3/4$

(c) Solve by Iterated Elimination of Strictly Dominated strategies (IESDS) the game in question (b) for all values of (x, y), by reducing the analysis to the few relevant cases.

(d) Check directly (without appealing to a theorem) that the set you find in all cases is the set of rationalizable strategies.

(e) Prove or provide a counterexample to the following statement:
Consider a two-player normal form game. Let A and B be two pure strategies of Player 1. Suppose that both A and B survive the IESDS procedure. Then any mixed strategy that attaches positive probability to both A and B, and zero to every other strategy, is a best reply to some mixed strategy of Player 2 .

290 Rahman

Question IV Fall 2020

Pelosian Barganing. Let $N = \{1, 2\}$ be a set of players, $A = \{S, T\} \times \{S, T\}$ a set of action profiles and $u(a) = (u_1(a), u_2(a))$ a payoff profile for each action profile $a \in A$, tabulated below:

	S	T
S	4, 4	1, 3
T	5, 2	α, 0

To begin with, assume that $\alpha = 0$ and consider the following bargaining problem. Let $F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq w^*\}$, where w^* is the maximum sum of payoffs achievable from the game above, be the set of feasible payoff profiles. Given a disagreement point $v \in F$, the Nash bargaining solution solves

$$\varphi(v) \in \arg\max_{x \in F} (x_1 - v_1)(x_2 - v_2)$$

(a) Find the Nash bargaining solution as a function of v.

(b) Compute the Nash bargaining solution when the disagreement point equals the payoff profile from the Nash equilibrium of the game above.

(c) Find a Nash equilibrium of the two-person game whose set of action profiles is A above and whose payoffs are $\varphi(u(a))$ for every $a \in A$. What is the profile of expected payoffs from this equilibrium?

(d) How do these payoffs change if $\alpha = -1$? What if $\alpha = -3$?

291 Werner

Question I January 2021

Consider two random variables $\tilde{y}$ and $\tilde{z}$ with given cumulative distribution functions on R and the same expectations $E(\tilde{y}) = E(\tilde{z})$

1. Show that, if $\tilde{y}$ is more risky than $\tilde{z}$, then $\text{var}(\tilde{y}) \geq \text{var}(\tilde{z})$, where var denotes the variance. Next, consider the optimal portfolio choice problem with one risky asset with return $\tilde{r}$ and a risk-free asset with return r_f. Suppose that the agent's von Neumann-Morgenstern (or Bernoulli) utility function is quadratic,

$$v(x) = \alpha x - \beta x^2 + \gamma$$

where $\alpha > 0$ and $\beta > 0$. Assume that $\frac{\alpha}{2\beta} > wr_f$, where $w > 0$ is the agent's initial wealth. Negative investment (i.e., short selling) is permitted for both assets.

2. Find the optimal investment in the risky asset.

3. Show that the optimal investment is strictly positive if and only if $E(\tilde{r}) > r_f$. Is the optimal investment an increasing or decreasing function of wealth w? Suppose that the return $\tilde{r}$ on the risky asset is changed to a more risky return $\tilde{r}'$ with the same expected value, that is with $E(\tilde{r}') = E(\tilde{r})$. Assume that $E(\tilde{r}) > r_f$

4. Prove that the optimal investment in the risky asset with more risky return $\tilde{r}'$ is smaller then the optimal investment with return $\tilde{r}$

292 Phelan

Question II January 2021

Consider a 2 good, 2 agent world. Good one denotes oranges and good two denotes orange juice. Each agent has utility function $u(c_1, c_2) = c_2$. Suppose agent A is endowed with 2 oranges and no orange juice. Agent B is endowed with no oranges or orange juice, but owns 100% of two identical firms each of which transforms oranges into orange juice according to the production function $f(x) = 2x$

1. Draw the production sets for each firm.

2. Define and find ALL competitive equilibrium (given these endowment and ownership specifications).

3. Redo the previous two parts assuming the first firm has production function $f(x) = 2x$ (as before), but the second firm has production function $f(x) = 2\sqrt{x}$.

293 Rustichini

Question III January 2021

Consider the simplified Poker game. There are a deck with three cards. The cards are ordered: A, B and $C.A$ beats B which beats C. There are two players, $P1$ and $P2$. The timeline is as follows:

- Two cards (x, y) are selected from the deck without replacement by nature; the cards are not shown to the players. Each player put one dollar in the pot, (the current total of money at stake). Card x is given to player $P1$, and card y to $P2$; each player sees the card he or she receives.
- $P1$ can pass or bet; if $P1$ passes, the game ends and $P2$ wins the pot. If $P1$ bets, he has to add one dollar to the pot, and the move goes to $P2$.
- $P2$ can fold or see. If $P2$ folds the game ends and $P1$ wins the pot. If $P2$ sees, $P2$ has to add one dollar to the pot. Then both player show the card and the player with the winning card among the two wins the pot.

Do the following:

1. Write the extensive form game, including monetary payoffs at the end nodes.
2. Write the pure strategies of $P1$ and $P2$. Are behavioral and mixed strategies equivalent in this game?
3. Write the normal form of the game. Apply iterated elimination of weakly dominated strategies. Apply iterated elimination of strictly dominated strategies.
4. Is there a pure strategy equilibrium? If not, provide a proof. Compute the Nash equilibrium of the game.

294 Rahman

Question IV January 2021

Consider a buyer and a seller of a single indivisible good. The seller's opportunity cost is 0 . The buyer has two possible types: θ_L and θ_H. The buyer knows her own type; the seller believes that θ_H has probability p and θ_L has probability $1-p$. The buyer's outside option, that is, the value to her of rejecting the seller upon learning θ, equals 0 regardless of her type. Each individual has quasilinear preferences over monetary transfers, as usual.

- Assume first that the buyer's valuation v is given by $v(\theta_L) = \frac{1}{3}$ and $v(\theta_H) = \frac{1}{2}$. Write down and solve the problem of designing a mechanism (report-contingent allocation and payment) that maximizes the seller's revenue subject to incentive compatibility and individual rationality.
- Suppose that the buyer's valuation is given by $v(\theta_H, \sigma) = \sigma$ and $v(\theta_L, \sigma) = \sigma^2$, where $\sigma \sim U[0,1]$ independently of θ. Neither the buyer nor the seller knows σ to begin with, but it is publicly revealed to both after the buyer has reported θ. Write down and solve the problem of designing a mechanism that maximizes the seller's revenue subject to incentive compatibility and individual rationality, where the allocation and payment and may depend on both θ and σ.
- Suppose that, in part (b) above, only the buyer observes σ. Is the allocation that helped solve the problem in (b) still incentive compatible when σ must be solicited by the seller? Why or why not? Show that the seller's revenue when only the buyer observes σ is weakly lower than the optimal revenue from part (b). Is it strictly lower? Why or why not?
- Suppose instead that, after reporting her type θ, the buyer privately learns only whether σ is less than some given threshold σ^* or greater-than-or-equal to it, and can report this to the seller as part of a trading mechanism. If the seller could choose σ^* together with a revenue-maximizing mechanism, what should the seller do? How does this compare with your answer to part (2)?