Math Appendix

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1 Introduction

1.1 Logic

Meaning	*Command*	*Notation*
Not	`\neg`	$\neg$
There exists	`\exists`	$\exists$
For all	`\forall`	$\forall$
Implies	`\implies`	$\Rightarrow$
Equivalent	`\iff`	$\iff$
And	`\land`	$\land$
Or	`\lor`	$\lor$
Defined as	`:=`	$:=$
Logical equivalence	`\equiv`	$\equiv$
Therefore	`\therefore`	$\therefore$
Because	`\because`	$\because$

Truth table

P	Q	$P \Rightarrow Q$	$\neg P$	$P \lor Q$	$P \land Q$
T	T	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	F
F	F	T	T	F	F

*These notes are intended to summarize the main concepts, definitions and results covered in the first year of micro, macro and metrics sequence for the Economics PhD of the University of Minnesota. The material is not my own. Please let me know of any errors that persist in the document. E-mail: pawel042@umn.edu .

Logical equivalences

Commutative	$p \wedge q \Longleftrightarrow q \wedge p$	$p \vee q \Longleftrightarrow q \vee p$
Associative	$(p \wedge q) \wedge r \Longleftrightarrow p \wedge (q \wedge r)$	$(p \vee q) \vee r \Longleftrightarrow p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \Longleftrightarrow p$	p $\vee F \Longleftrightarrow p$
Negation	$p \vee \sim p \Longleftrightarrow T$	p $\wedge \sim p \Longleftrightarrow F$
Double Negative	$\sim (\sim p) \Longleftrightarrow p$	
Idempotent	$p \wedge p \Longleftrightarrow p$	$p \vee p \Longleftrightarrow p$
Universal Bound	$p \vee T \Longleftrightarrow T$	p $\wedge F \Longleftrightarrow F$
De Morgan's	$\sim (p \wedge q) \Longleftrightarrow (\sim p) \vee (\sim q)$	$\sim (p \vee q) \Longleftrightarrow (\sim p) \wedge (\sim q)$
Absorption	$p \vee (p \wedge q) \Longleftrightarrow p$	$p \wedge (p \vee q) \Longleftrightarrow p$
Conditional	$(p \Longrightarrow q) \Longleftrightarrow (\sim p \vee q)$	$\sim (p \Longrightarrow q) \Longleftrightarrow (p \wedge \sim q)$

1.2 Greek letters

Command	*Notation*	*Command*	*Notation*
`\alpha`	α	`\tau`	τ
`\beta`	β	`\theta`	θ
`\chi`	χ	`\upsilon`	υ
`\delta`	δ	`\xi`	ξ
`\epsilon`	ϵ	`\zeta`	ζ
`\varepsilon`	ε	`\Delta`	Δ
`\eta`	η	`\Gamma`	Γ
`\gamma`	γ	`\Lambda`	Λ
`\iota`	ι	`\Omega`	Ω
`\kappa`	κ	`\Phi`	Φ
`\lambda`	λ	`\Pi`	Π
`\mu`	μ	`\Psi`	Ψ
`\nu`	ν	`\Sigma`	Σ
`\omega`	ω	`\Theta`	Θ
`\phi`	ϕ	`\Upsilon`	Υ
`\varphi`	φ	`\Xi`	Ξ
`\pi`	π	`\aleph`	$\aleph$
`\psi`	ψ	`\beth`	$\beth$
`\rho`	ρ	`\daleth`	$\daleth$
`\sigma`	σ	`\gimel`	$\gimel$

1.3 Proving Things

Let P and Q be two statements.

- We say that "P implies Q ", or "if P then Q", and note $P \Rightarrow Q$, if Q is true when P is true.
- We say that P is a **sufficient condition** for Q, and Q is a **necessary condition** for P.
- $P \Rightarrow Q$ and $Q \Rightarrow P$ are two very different statements. We call one the converse of the other.

- We say that "P and Q are equivalent", or "P if and only if (iff) $\iff$ Q", and note $P \Leftrightarrow Q$, if P implies Q and Q implies P.
- The implication $P \Rightarrow Q$ is equivalent to its contrapositive $not(\text{Q}) \Rightarrow not(\text{P})$.
- If P is a statement, the negation of P, $not(\text{P})$, is true when P is false and false when P is true. Notice that the negation "reverses the quantifiers":

$$not(\exists x \in X : \text{P}(x)) \text{ is equivalent to } \forall x \in X : not(P(x))$$
$$not(\forall x \in X : \text{P}(x)) \text{ is equivalent to } \exists x \in X : not(P(x))$$

Types of proofs:

1. To prove that " $\forall x : P(x)$ " is false, we look for a **counter-example**. Exercises are sometimes phrased "Provide a proof if it is true, and a counter-example if it is false"; but "prove your answer" is an equivalent requirement, as providing a counter-example proves the negation. If only such statement is indeed false.
2. Sometimes, it is easier to prove the implication $P \Rightarrow Q$ by proving its contradiction $not(\text{Q}) \Rightarrow not(\text{P})$. We call it a **proof by contradiction** or **reductio ad absurdum** or **indirect proof**. A proof by contradiction is sometimes very helpful, as standard methods of proofs do not work. To prove P by contradiction, we assume not(P) and derive true statements until we end up proving that a statement we know to be true is false (this can be any statement in the mathematical edifice).
3. An **equivalence** consists of two implications. To show an equivalence, we show both implications.
4. **Proving a $\forall$**. To prove a statement of the form " $\forall x \in X : P(x)$ ", we fix an $x \in X$, and prove $P(x)$, being careful to use a reasoning that applies to any $x \in X$.
5. **Proving a $\exists$**. Proving a statement of the form " $\exists x \in X : P(x)$ " is more difficult. We need to find point at an x that works-and satisfies P(x)
6. **Proving uniqueness** To show that " $\exists! x : P(x)$", show existence and uniqueness separately. To show uniqueness, assume there exist two x such that $P(x)$ and show that they are equal.
7. **Induction** or **proof by induction**. We want to show that

$$\forall n \in \mathbb{N} : P(n)$$

   To prove this by induction, we prove two things:

   (a) **The base case**: we prove $P(0)$ (or $P(N)$ more generally).

   (b) **The inductive step**: we prove that $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$ (or for all $n \geq N$ more generally).

1.4 General notation

- $x \in A$ - x is an element of set A
- $B \subseteq C$ - set B is a subset of set C
- $B = C \iff B \subseteq C$ and $C \subseteq B$
- $\mathbb{N} = \{0, 1, 2 \dots\}$ natural numbers
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2 \dots\}$ integers
- $\mathbb{Q} = \{q : \quad \exists a, b \in \mathbb{Z} \quad q = \frac{a}{b}\}$ rational number
- $\mathbb{R}$ real numbers
- $\mathbb{R}^n := \{x = (x_1, \dots, x_i \dots, x_n) : x_i \in \mathbb{R}, \quad \forall i = 1, \dots, n\}$ n- dimensional real Euclidean space
- $\mathbb{R}^n_+ := \{x = (x_1, \dots, x_i \dots, x_n) : x_i \geq 0, \quad \forall i = 1, \dots, n, \quad \text{and} \quad x \neq 0\}$
- For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ we denote

$$x \geq y \iff x_i \geq y_i, \quad \forall i = 1, \dots, n$$
$$x > y \iff x \geq y \quad and \quad x \neq y$$
$$x \gg y \iff x_i > y_i, \forall i = 1, \dots, n$$

- Cartesian Product of a finite collection of sets $E_1, E_2, \dots, E_n$ is the set of ordered n -tuples $E_1 \times E_2 \times \dots \times E_n = \{(x_1, x_2, \dots, x_n) : x_j \in E_j \quad \forall j = 1, 2, \dots n\}$
- $x \cdot y$ or $< x, y >$ denotes the scalar product of x and $y \in \mathbb{R}^n$ so $x \cdot y = \sum_{i=1}^{n} x_i y_i$
- A matrix $A \in \mathbb{R}^{m \times n}$ is a matrix with m rows and n columns with (i, j) entry a_{ij}
- A is a matrix with m rows and n columns and B is a matrix with n rows and l columns, AB denotes the matrix product of A and B.
- H is a $n \times n$ matrix, tr (H) denotes the **trace** of H, $det(H)$ denotes the **determinant** of H, and $cof(H)$ denotes **cofactor** of H
- $x \in \mathbb{R}^n$ is treated as a row matrix so $1 \times n$.
- $e_i = (0, \dots, 1, \dots, 0)$- ith standard coordinate vector
- x^T denotes the transpose of $x \in \mathbb{R}^n$, x^T is treated as a column matrix so $n \times 1$.
- $f : X \to \mathbb{R}$ so f is a function from (open set) $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$
- $f \in \mathcal{C}^p$ - function f is $\mathcal{C}^p$ class- derivatives up to order p are continuous
- $f \in \mathcal{C}^0$- function f is continuous

2 Binary relations

Definition 1. *Assumptions on binary relations ($R :\succeq, P :\succ, I :\sim$)*

a ***reflexive*** *:* $\forall_a \quad aRa$

b ***irreflexive****:* $\forall_a \quad \neg(aRa)$

c ***symmetric****:* $\forall_{a,b} \quad aRb \iff bRa$

d ***asymmetric****:* $\forall_{a,b} \quad aRb \iff \neg(bRa)$

e ***antisymmetric****:* $\forall_{a,b} \quad aRb \wedge bRa \Rightarrow a = b$

f ***complete****:* $\forall_{a,b} \quad aRb \vee bRa$

g ***transitive*** $\forall_{a,b,c} \quad aRb \wedge bRc \Rightarrow aRc$

h ***negative transitive*** $\forall_{a,b,c} \quad \neg(aRb) \wedge \neg(bRc) \Rightarrow \neg(aRc)$

Definition 2. *Main categories of binary relations*

a ***(Weak) Preorder aka Preference Relation****- Reflexive, Transitive*

b ***Equivalence Relation****- Reflexive, Symmetric, Transitive*

c ***Strict partial order*** *-Asymmetric, Transitive*

d ***Partial Order****- Reflexive, Antisymmetric, Transitive*

e ***Total (or Linear) Order****- Antisymmetric, Complete, Transitive*

2.1 Monotonicity and Nonsatiation

Assume that relation $R := \succeq$ is a preorder.

Definition 3. $\succeq$ *is* ***weakly monotone*** *on a set* X *if* $\forall x, y \in X$,

$$x \geq y \Rightarrow x \succeq y$$

Definition 4. $\succeq$ *is* ***monotone*** *on a set* X *if* $\forall x, y \in X$,

$$x \gg y \Rightarrow x \succ y$$

.

Definition 5. $\succeq$ *is* ***strongly monotone*** *on a set* X *if* $\forall x, y \in X$,

$$(x \geq y \wedge x \neq y) \Rightarrow x \succ y$$

.

Definition 6. $\succeq$ *is* ***locally nonsatiated*** *on a set* X *if*

$$\forall x \in X \text{ and } \forall \epsilon > 0 \quad \Rightarrow \quad \exists y \in X \ni \| x - y \| < \epsilon \text{ and } y \succ x$$

2.2 Convexity

Definition 7. *$\succeq$ is **weakly convex** on a set X if $\forall x, y \in X, \forall \lambda \in (0,1)$,*

$$x \succeq y \Rightarrow \lambda x + (1-\lambda)y \succeq y$$

Definition 8. *$\succeq$ is **convex** on a set X if $\forall x, y \in X, \forall \lambda \in (0,1)$,*

$$x \succ y \Rightarrow \lambda x + (1-\lambda)y \succ y$$

Definition 9. *$\succeq$ is **strongly/strictly convex** on a set X if $\forall x, y \in X, \forall \lambda \in (0,1)$,*

$$x \sim y \wedge x \neq y \Rightarrow \lambda x + (1-\lambda)y \succ y$$

2.3 Continuity

Definition 10 (Sequential definition/ weak continuity). *A preorder $\succeq$ is continuous on a set X if $\forall \{x_n\}, \{y_n\} \subseteq X$,*

$$\forall n \in \mathbb{N}, (x_n \succeq y_n) \wedge (x_n \to x) \wedge (y_n \to y) \Rightarrow x \succeq y$$

Definition 11 (Set definition/ strong continuity). *A preorder $\succeq$ is continuous on a set X if $\forall x \in X$, the upper contour set $U(x) = \{y \in X : y \succeq x\}$ and the lower contour set $L(x) = \{y \in X : x \succeq y\}$ are closed in X.*

3 Real analysis

3.1 Properties of $\mathbb{R}$

Definition 12 (The Algebraic Properties of $\mathbb{R}$). *On $\mathbb{R}$ there are two binary operations: $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:*

- a *(commutative property of addition)* $a + b = b + a$ *for all* a, b *in* $\mathbb{R}$
- b *(associative property of addition)* $(a + b) + c = a + (b + c)$ *for all* a, b, c *in* $\mathbb{R}$
- c *(existence of a zero element)* $\exists 0 \in \mathbb{R}$ *s.t.* $\forall a \in \mathbb{R}, 0 + a = a$ *and* $a + 0 = a$
- d *(existence of negative elements)* $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}$ *s.t.* $a + (-a) = 0$ *and* $(-a) + a = 0$
- e *(commutative property of multiplication)* $a \cdot b = b \cdot a$ *for all* $a, b \in \mathbb{R}$
- f *(associative property of multiplication)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for all* a, b, c *in* $\mathbb{R}$
- g *(existence of a unit element)* $\exists 1 \in \mathbb{R}, 1 \neq 0$ *s.t.* $\forall a \in \mathbb{R}, 1 \cdot a = a$ *and* $a \cdot 1 = a$
- h *(existence of reciprocals)* $\forall a \neq 0, a \in \mathbb{R}$, *exists an element* $1/a \in \mathbb{R}$ *s.t.* $a \cdot (1/a) = 1$ *and* $(1/a) \cdot a = 1$
- i *(distributive property of multiplication over addition)* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ *and* $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ *for all* a, b, c *in* $\mathbb{R}$

Definition 13 (Bounded Set). *Let* $S \subseteq \mathbb{R}, S \neq \varnothing, S$ *is said to be*

a ***bounded above*** *if there exists a number* $u \in \mathbb{R}$*, s.t.* $\forall s \in S, s \leq u$*. Such* u *is called an u**upper bound** of* S*.*

b ***bounded below*** *if there exists* $w \in \mathbb{R}$*, s.t.* $\forall s \in S, s \geq w$*. Such* w *is called a lower bound of* S*.*

c *A set is said to be **bounded** if it both bounded above and below; unbounded if not bounded.*

Definition 14 (Sup and Inf). *Let* $S \subseteq \mathbb{R}, S \neq \varnothing$

1. *If* S *is bounded above, then a number* u *is said to be supremum (or a least upper bound) of* S *if:*
   (a) u *is an upper bound of* S
   (b) *For any upper bound of* $\mathrm{S} : v, \quad u \leq v$
2. *If* S *is bounded below, then a number* w *is said to be infimum (or a greatest lower bound) of* S *if:*
   (a) w *is an lower bound of* S
   (b) *For any lower bound of* $\mathrm{S} : t, \quad t \geq w$

Lemma 1. *A set* $S \subseteq \mathbb{R}$ *has only one supremum (same for infimum).*

Definition 15 (Supremum). $\sup S \iff \forall_{\epsilon>0} \exists_{a \in S} \sup S - \epsilon < a \leq \sup S$

Lemma 2. *Every nonempty set of real numbers that has an upper bound also has a supremum in* $\mathbb{R}$

Lemma 3 (Archimedean Property). *If* $x \in \mathbb{R}$*, then there exists* $n_x \in \mathbb{N}$ *s.t.* $x \leq n_x$*.*

Lemma 4 (The Density Theorem). *If* $x, y \in \mathbb{R}, x < y$*, then there exists a rational number* $r \in \mathbb{Q}$ *s.t.* $x < r < y$*.*

Definition 16 (Nested). *A sequence of intervals* $I_n, n \in \mathbb{N}$ *is nested if the following chain of inclusions holds:*

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

Lemma 5 (Nested Intervals Property). *If* $I_n = [a_n, b_n], n \in \mathbb{N}$*, is a nested sequence of closed bounded intervals, then there exists a number* $\xi \in \mathbb{R}$ *such that* $\xi \in I_n$ *for all* $n \in \mathbb{N}$

3.2 Normed vector space $\mathbb{R}^n$

From now on $x \in \mathbb{R}^n$

Definition 17 (Norm). $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ *such that*

1. $\|x\| \geq 0 \wedge \|x\| = 0$ *iff* $x = 0$ ***positive definiteness***
2. $\forall_{\alpha\in\mathbb{R}}\forall_{x\in\mathbb{R}^n} \quad \|\alpha x\| = |\alpha|\|x\|$ ***homogenity***
3. $\|x + y\| \leq \|x\| + \|y\| \quad \forall_{x,y\in\mathbb{R}^n}$ $\triangle$ ***-inequality***

Examples

- Absolute Value of a real number $x \in \mathbb{R}$:

$$|x| := \begin{cases} x & if x > 0 \\ 0 & if x = 0 \\ -x & if x < 0 \end{cases}$$

- Euclidean norm: $\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$
- 1-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- L^1 norm: $\|f\|_{L^1} = \int |f(x)|dx$
- sup norm: $\|x\|_\infty = \sup_{i\in\{1,\dots,n\}} |x_i|$

Definition 18 (Vector Space). *A set E is called vector space, if it endowed with two operations (or E is closed under both operations):*

- *Addition:* $E \times E \to E$
- *Scalar multiplication:* $\mathbb{R} \times E \to E$

s.t. $\forall x, y, z \in E, a, b \in \mathbb{R}$

a *(Commutativity)* $x + y = y + x$

b *(Associativity)* $x + (y + z) = (x + y) + z$

c *(Existence of zero)* $\exists 0 \in E$ *s.t.* $\forall x \in E, x + 0 = x$

d *(Existence of additive inverse)* $\forall x \in E, \exists(-x) \in E$ *s.t.* $x + (-x) = 0$

e *(Associative)* $(ab)x = a(bx)$

f *(Distributive)* $a(x + y) = ax + ay, (a + b)x = ax + bx$

Definition 19 (Normed Vector space). *A normed vector space is a vector space E endowed in a norm (which is a function)* $\|\cdot\| : E \to \mathbb{R}$

Theorem 1 (Cauchy-Schwarz inequality).

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Examples

- $cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}X \cdot \mathbb{E}Y \leq \sqrt{VAR(X)}\sqrt{VAR(Y)}$
- $\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$

3.3 Topology on $\mathbb{R}^n$

Definition 20 (Convergence).

$$x_n \to x \iff \forall_{\epsilon>0}\exists_N\forall_{k\geq N} \quad x_k \in B_\epsilon(a)$$

Definition 21 (Cauchy sequence). $\forall_{\epsilon>0}\exists_N\forall_{m,n>N} \quad ||x_n - x_m|| < \epsilon$

Definition 22 (Open ball). $B_\epsilon(a) = \{x \in \mathbb{R}^n : \|x-a\| < \epsilon\}$.

Definition 23 (Open set). *$V \subseteq \mathbb{R}^n$ is open iff $\forall_{x\in V}\exists_{\epsilon>0} B_\epsilon(x) \subseteq V$*

Lemma 6. *$x_k \to a$ as $k \to \infty$ if and only if for every open set V that contains a there is an $N \in \mathbb{N}$ such that $k \geq N$ implies $x_k \in V$*

Definition 24 (Closed set). *$E \subseteq \mathbb{R}^n$ is closed if $E^c = \mathbb{R}^n \backslash E$ is open.*

Lemma 7. *E is closed if and only if E contains all its limit points; i.e., $x_k \in E$ and $x_k \to x$ imply $x \in E$*

Definition 25 (Interior of a set). $E^\circ = \bigcup\{V : V \subseteq E \wedge V \in \tau(\mathbb{R}^n)\}$

Definition 26 (Closure of a set). $\bar{E} = cl(E) = \cap\{V : E \subseteq V \wedge V \in \mathcal{F}(\mathbb{R}^n)\}$, *where $\mathcal{F}(\mathbb{R}^n)$ is family of closed sets.*

Definition 27 (Boundary). $\partial E = \{x \in \mathbb{R}^n \mid \forall_\epsilon \quad B_\epsilon(x) \cap E \neq \emptyset \quad \wedge \quad B_\epsilon(x) \cap E^c \neq \emptyset\}$

Lemma 8.
- $E^\circ \subseteq E \subseteq \bar{E}$
- *$E^\circ = E$ iff E is open*
- *$\bar{E} = E$ iff E is closed*
- $\partial E = \bar{E}/E^\circ$

Definition 28 (Topological space). $(X, \tau))$.

1. $\emptyset, X \in \tau$
2. *if $\{v_\alpha\} \in \tau \Rightarrow \bigcup_{\alpha\in I} v_\alpha \in \tau$ any collections*
3. *if $\{v_\alpha\} \in \tau \Rightarrow \bigcap_{\alpha\in I} v_\alpha \in \tau$ finite*

Definition 29 (Connected set X). $\#_{U,V\in\tau} U \cup V = X \wedge U \cap V = \emptyset$

Definition 30 (Complete set X). *Every Cauchy sequence of points in X has a limit that is also in X*

Metric generating completeness not topology is important (we skip metric here)

Definition 31 (Open covering). *of E is a collection of sets $\{V_\alpha\}_{\alpha\in A}$ such that each V_α is open and*

$$E \subseteq \bigcup_{\alpha\in A} V_\alpha$$

Definition 32 (Compactness). *. $E \subseteq \bigcup_{\alpha\in I} v_{\alpha\alpha} \in \tau.E$ is compact if for every open covering of E it has always finite subcover.*

Lemma 9. *Each metric generating topology of compact space is complete. Every complete and totally bounded space is compact.*

Theorem 2 (Heine-Borel). $E \subseteq \mathbb{R}^n$ *is compact* $\Longleftrightarrow E$ *closed and bounded.*

Lemma 10. *($\exists$ of convergent subsequence). If* $E \subseteq \mathbb{R}^n$ *is compact,* $\{x_n\} \subseteq E \Rightarrow \exists_{x_{n_k}} x_{n_k} \to x$

Definition 33 (Continuity in topological space). .

$$f \in \mathcal{C}^0 \text{ on } X \Longleftrightarrow \forall_{V \in \tau(X)} f^{-1}(V) \in \tau \qquad (\Longleftrightarrow \forall_{U \in \mathcal{F}(x)} f^{-1}(U) \in \mathcal{F})$$

Definition 34 (Continuity at a point). $f : \Phi \to X$ *is continuous at* $\Theta \Longleftrightarrow \forall_{open V \subseteq X : f(\Theta) \in V} \exists_{open U \subseteq \Phi} \Theta \in U$

3.4 Continuity and Convergence

3.4.1 Sequences

Definition 35 (Convergence).

$$\lim_{n \to \infty} x_n \to x \Longleftrightarrow \forall_\mu \exists_N \forall_{n \geq N} |x_n - x| < \epsilon$$

divergent sequence: $(x_n \to \pm\infty) \Longleftrightarrow \forall_\mu \exists_N \forall_{n \geq N} x_n > \mu \, (x_n < \mu)$

Lemma 11 (Uniqueness of Limits). *A sequence in* $\mathbb{R}$ *can have at most one limit.*

Lemma 12. *A convergent sequence of real numbers is bounded (but not conversely).*

Theorem 3 (Squeeze Theorem). *aka the between theorem aka the sandwich theorem. Suppose* $\{x_n\}, \{y_n\}, \{z_n\}$ *are sequences of real numbers s.t.*

$$x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$$

and $\lim(x_n) = \lim(z_n)$*, then* y_n *is convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

Lemma 13. *Every convergent sequence is bounded.*

Definition 36 (Monotone sequence). $\{a_n\}$ *is* ***increasing*** *if*

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \ldots$$

is ***decreasing*** *if*

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \ldots$$

is monotone if either increasing or decreasing.

Theorem 4 (Monotone convergence). *If* x_n *is monotone and bounded* $\Rightarrow x_n \to x < +\infty$

Definition 37 (Subsequence). *let* $n_1 < n_2 < \cdots < n_k < \ldots$ *be a strictly increasing sequence from* $\mathbb{N}$*. The sequence* $\{a_{n_k}\}$ *given by* $\{a_{n_1}, a_{n_2} \ldots, a_{n_k}, \ldots\}$ *is called a subsequence of* $\{a_n\}$

Lemma 14. *If a sequence $\{a_n\}$ converges to a real number a, then any subsequence of $\{a_n\}$ converges to a.*

Theorem 5 (W.Sierpinski). *If $\{a_n\}$ is a sequence of real numbers, then there exists a subsequence of $\{a_n\}$ that is monotone.*

Theorem 6 (Bolzano-Weierstrass). *Every bounded sequence has convergent subsequence*

Definition 38 (Cauchy sequence). . *$\{x_n\}$ is Cauchy $\Longleftrightarrow$*

$$\forall_{\epsilon>0}\exists_N\forall_{n,m\geq N} \mid x_n - x_m \mid < \epsilon$$

Lemma 15. *If $x_n \to x \Rightarrow \{x_n\}$ is Cauchy*

Theorem 7 (Cauchy Convergence Criterion on $\mathbb{R}$). *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Definition 39 (limsup, liminf). .

$$\limsup x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$
$$\liminf x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

Lemma 16. $x_n \to x \Longleftrightarrow \limsup x_n = f x_n$

3.4.2 Functions

Definition 40 (Limit of function).

$$\lim_{x\to a} f(x) = y \Longleftrightarrow \forall_{\epsilon>0}\exists_{\delta>0}|x-a| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$$

Definition 41 (Right/left limits). .
Right

$$y = \lim_{x\to a^+} f(x) \Longleftrightarrow \forall_{\epsilon>0}\exists_{\rho>0} a < x < a+\rho \Rightarrow |f(x)-y| < \epsilon$$

Left

$$z = \lim_{x\to a^-} f(x) \Longleftrightarrow \forall_{\epsilon>0}\exists_{\rho>0} a-\rho < x < a \Rightarrow |f(x)-z| < \epsilon$$

Definition 42 (Continuity). . *$f : E \to \mathbb{R}$ is continuous at $x \in E \Longleftrightarrow$*

$$x_n \to x \Rightarrow f(x_n) \to f(x)$$

$$\forall_{\epsilon>0} \quad \exists_{\delta>0} \quad ||x_n - x|| < \delta \Rightarrow \quad ||f(x_n) - f(x)|| < \epsilon$$

f is continuous on X if f is continuous at every point $y \in X$

Lemma 17. *Equivalently*

- *f is continuous at $y \in X$ if and only if for every open ball J of center $f(y)$ there exists an open ball B of center y such that $f(B \cap X) \subseteq J$*
- *f is continuous at $y \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x-y\| < \delta$ and $x \in X \Longrightarrow |f(x) - f(y)| < \varepsilon$*

Lemma 18 (Sequentially continuous function). *f is continuous at $y \in X$ if and only if f is sequentially continuous at y, that is, for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to y$, we have that*

$$f(x_n) \to f(y)$$

Theorem 8. *Let $E \subseteq \mathbb{R}$, f, g and $h : E \to \mathbb{R}$, let $c \in \mathbb{R}$ be a cluster point of $E, b \in \mathbb{R}$. If $f(x)$ and $g(x)$ converge as $x \to c$, then*

1. $\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(fg)(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$
3. $\lim_{x\to c}(\alpha f)(x) = \alpha(\lim_{x\to c} f(x))$
4. $\lim_{x\to c}(f/g)(x) = \lim_{x\to c} f(x)/\lim_{x\to c} g(x)$ *(with $g(x)$'s limit not equal to 0.)*
5. *If $a \leq f(x) \leq b$ for all $x \in E\backslash c$, then $a \leq \lim_{x\to c} f(x) \leq b$*
6. *(Squeeze Theorem) If $g(x) \leq h(x) \leq g(x)$ for all $x \in E\backslash\{c\}$ and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = y$ then $\lim_{x\to c} h(x) = y$*
7. *If $|g(x)| < M$ for all $x \in E\backslash\{c\}$ and $f(x) \to 0$ as $x \to c$ then $\lim_{x\to c} f(x)g(x) = 0$*

Lemma 19 (Boundedness Lemma). *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.*

Theorem 9 (Extreme value theorem - Weierstrass). . *If $H \subseteq \mathbb{R}^n$ compact, $f : H \to, f \in \mathcal{C}^0$ then*

$$\exists y, \underline{x} \quad f(y) = \sup_{x\in H} f(x) \quad \wedge \quad f(\underline{x}) = \inf_{x\in H} f(x)$$

Theorem 10 (Intermediate value theorem, Darboux). *$f : I \to \mathbb{R}, a, b \in I, a < b, y_0 \in (f(a), f(b)) \Rightarrow \exists_{x_0\in(a,b)} f(x_0) = y_0$*

Lemma 20. *Let $f : I \to \mathbb{R}$ be continuous on I.*

a *If I is a closed bounded interval, then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded interval.*

b *If I is an interval, then the set $f(I)$ is an interval.*

c *If K is a compact subset of $\mathbb{R}$, then $f(K)$ is compact.*

Definition 43 (Uniform continuity). $\forall x_{1,x_2} \forall_{\epsilon>0} \exists_{\rho>0} |x_1 - x_2| < \rho \Rightarrow \|f(x_1) - f(x_2)| < \epsilon$ *then $f(x_n)$ is Cauchy*

Theorem 11 (Uniform Continuity Theorem). *I compact interval and $f : I \to$ R be continuous on I. Then f is uniformly continuous on I.*

Theorem 12 (Easy version of Tietze). *A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$*

Definition 44 (Pointwise convergence).

$$\forall_x f(x) = \lim_{n\to\infty} f_n(x) \iff \forall_x \forall_{\epsilon>0} \exists_N \forall_{n>N} |f_n(x) - f(x)| < \epsilon$$

Definition 45 (Uniform convergence $\rightrightarrows$).

$$\forall_{\epsilon>0} \exists_N \forall_{n>N} \forall_{x\in E} |f_n(x) - f(x)| < \epsilon \iff \forall_{\epsilon>0} \exists_N \forall_{m,n>N} \forall_{x\in E} |f_n(x) - f_m(x)| < \epsilon$$

Theorem 13 (Lebesgue Dominated convergence theorem).

$$\forall_n |f_n| < M, f_n \rightrightarrows f \Rightarrow f < M$$

Definition 46. *f is **weakly increasing (or non-decreasing)** on X if for all x and y in X*

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

Definition 47. *f is **increasing** on X if for all x and y in X*

$$x \ll y \Longrightarrow f(x) < f(y)$$

Definition 48. *f is **strictly increasing** on X if for all x and y in X,*

$$x < y \Longrightarrow f(x) < f(y)$$

Almost all properties above hold for $\mathbb{R}^n$

3.5 Differentiability

Definition 49 (Partial Derivative). *: Denote a function $f : \{x_1\} \times \cdots \times \{x_{j1}\} \times [a, b] \times \{x_{j+t}\} \times \cdots \times \{x_n\} \to \mathbb{R}$ as*

$$g(t) := f(x_1, \ldots, x_{j_1}, t, x_{j+1}, x_n), \quad t \in [a, b]$$

If g is differentiable at some $t_0 \in [a, b]$, then the partial derivative of f at $(x_1, \ldots, x_{j-1}, t_0, x_{j+1}, \ldots, x_n)$ with respect to x_j is defined by

$$f_j(x_1, \ldots, x_{j-1}, t_0, x_{j+1}, x_m) = \frac{\partial f}{\partial x_j}(x_1, \ldots, x_{j-1}, t_0, x_{t+1}, \ldots, t_m)$$

The partial derivative exists at a if and only if

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0} \frac{f(a + he_j) - f(a)}{h}$$

Similarly, for a vector-valued functions, the partial derivative at a is defined as

$$\frac{\partial f}{\partial x_j}(a) = \left(\frac{\partial f_1}{\partial x_j}(a), \ldots, \frac{\partial f_m}{\partial x_j}(a)\right)$$

Higher-order partial derivatives are defined by iteration. For example, a second partial derivative respect to x_j and x_k is defined by

$$f_{j,k} := \frac{\partial^2 f}{\partial x_k \partial x_j} := \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)$$

Definition 50. *$X \subseteq \mathbb{R}^n$ is an open set, f is a function from X to $\mathbb{R}$ and $x \in X$*

$$\nabla f(x) := \left(\frac{\partial f}{\partial x^1}(x), \ldots, \frac{\partial f}{\partial x^h}(x), \ldots, \frac{\partial f}{\partial x^n}(x)\right)$$

denotes the ***gradient*** *of f at x,*

Definition 51 (Frechet Differentiable function). *f is differentiable at $y \in X$ if*
1. all the partial derivatives of f at y exist,
2. there exists a function E_y defined in some open ball $B(0, \varepsilon) \subseteq \mathbb{R}^n$ such that for every $u \in B(0, \varepsilon)$

$$f(y+u) = f(y) + \nabla f(y) \cdot u + \|u\| E_y(u) \quad \text{where} \quad \lim_{u \to 0} E_y(u) = 0$$

f is differentiable on X if f is differentiable at every point $y \in X$.

Lemma 21. *If f is differentiable at y, then f is continuous at y*

Definition 52 (Gateaux differentiable- Directional derivative). *Let $v \in \mathbb{R}^n, v \neq 0$. The directional derivative $D_v f(y)$ of f at $y \in X$ in the direction v is defined as*

$$\lim_{t \to 0^+} \frac{f(y+tv) - f(y)}{t}$$

if this limit exists and it is finite.

Lemma 22 (Differentiable function/Directional derivative). *If f is differentiable at $y \in X$, then for every $v \in \mathbb{R}^n$ with $v \neq 0$*

$$D_v f(y) = \nabla f(y) \cdot v$$

Theorem 14. *Let $I \subset \mathbb{R}, c \in I$, and let $f, g : I \to \mathbb{R}$ be functions that are differentiable at c, $\alpha \in \mathbb{R}$. Then the following equation are also differentiable at c*

1. *αf, with $(\alpha f)' = \alpha f'$*
2. *$f + g$, with $(f+g)'(c) = f'(c) + g'(c)$*
3. *$f^n(c)$, with $(f^n)'(c) = n(f(c))^{n-1} f'(c)$*
4. *(Product Rule) fg, with $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$*
5. *(Quotient Rule) f/g, with (if $g(c) \neq 0$)*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Theorem 15 (Carathéodory's Theorem). *Let $f : I \to \mathbb{R}, c \in I$. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and satisfies*

$$f(x) - f(c) = \varphi(x)(x - c)$$

for $x \in I$ In this case, we have $\varphi(c) = f'(c)$

Theorem 16 (Chain Rule). *Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and (11)*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Theorem 17 (Interior Extremum Theorem). *Let c be an interior point of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$*

Theorem 18 (Roll's Theorem). *Suppose a continuous function $f : [a, b] \to \mathbb{R}$ has derivative f' exists at every point in (a, b) and $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a, b)$ s.t. $f'(c) = 0$*

Theorem 19 (L'Hospital's Rules). *Let $-\infty \leq a < b \leq \infty$, and let f, g differentiable on (a, b) and $g' \neq 0 \forall x \in (a, b)$.*

- *If*

  $$\lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x)$$

  then

  $$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$$

- *If $\lim_{x \to a+} g(x) = \pm\infty$, then*

  $$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$$

Left-hand limits and two-sided limits are treated in exactly the same way.

Definition 53. *$D^2 f(x) = H(x)$ denotes the* ***Hessian matrix*** *of f at x*

Definition 54. *$X \subseteq \mathbb{R}^n$ is an open set, $g := (g_1, \ldots, g_j, \ldots, g_m)$ is a mapping from X to $\mathbb{R}^m$ and $x \in X$*

$$\mathrm{J}g(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x^1}(x) & \cdots & \frac{\partial g_1}{\partial x^h}(x) & \cdots & \frac{\partial g_1}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_j}{\partial x^1}(x) & \cdots & \frac{\partial g_j}{\partial x^h}(x) & \cdots & \frac{\partial g_j}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x^1}(x) & \cdots & \frac{\partial g_m}{\partial x^h}(x) & \cdots & \frac{\partial g_m}{\partial x^n}(x) \end{bmatrix}_{m \times n} = \begin{bmatrix} \nabla g_1(x) \\ \vdots \\ \nabla g_j(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}_{m \times n}$$

denotes the ***Jacobian matrix*** *of g at x.*

3.6 Powerful theorems of analysis

Theorem 20 (Inverse Function Theorem). *Let V be open in $\mathbb{R}^n$ and $f : V \to \mathbb{R}^n$ be $\mathcal{C}^1$ on V. If $\Delta_f(a) \neq 0$ for some $a \in V$, then there exists an open set W containing a such that*

- *f is $1-1$ on W*
- *f^{-1} is $\mathcal{C}^1$ on $f(W)$, and*
- *for each $y \in f(W)$*

$$D\left(f^{-1}\right)(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1}$$

Notation: $[\cdot]^{-1}$ represents matrix inversion, $\Delta_f(a) = det(Df(a))$ (the Jacobian of f at a)

Theorem 21 (Mean Value Theorem on $\mathbb{R}^n$). *Let $V \subseteq \mathbb{R}^n$ be open and convex, and let $f : V \to \mathbb{R}$ be a function that is differentiable everywhere on V. Then, for any $a, b \in V$, there is $\lambda \in (0,1)$ such that*

$$f(b) - f(a) = Df((1-\lambda)a + \lambda b) \cdot (b-a)$$

Notation: $L(a,b) := \{(1-t)\mathbf{a} + tb : t \in [0,1)\}$ is called line segment

Theorem 22 (Taylor Theorem on $\mathbb{R}^n$). *Let $p \in \mathbb{N}$, let V be open in $\mathbb{R}^n$, let $x, a \in V$, and suppose that $f : V \to \mathbb{R}$. If the pth total differential of f exists on V and $L(x;a) \subseteq V$, then there is a point $c \in L(x,a)$, $h := x - a$ such that*

$$f(x) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(a;h) + \frac{1}{p!} D^{(p)} f(c,h)$$

Theorem 23 (Implicit Function Theorem). *Let $F : S \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function, where S is open. Let (x^*, y^*) be a point in S such that $DF_y(x^*, y^*)$ is invertible, and let $F(x^*, y^*) = c$. Then, there is a neighborhood $U \subseteq \mathbb{R}^m$ of x^* and a $\mathcal{C}^1$ function $g : U \to \mathbb{R}^n$ such that*

- *$(x, g(x)) \in S, \forall x \in U$*
- *$g(x^*) = y^*$*
- *$F(x, g(x)) \equiv c, \forall x \in U$*
- *$Dg(x) = (DF_y(x,y))^{-1} \cdot DF_x(x,y)$*

3.7 Concavity and quasi-concavity

In this section, we assume that C is a convex subset of $\mathbb{R}^n$ and f is a function from C to $\mathbb{R}$.

Definition 55 (Convex set). *A set C is convex $\Longleftrightarrow \forall_{x,y \in \mathrm{C}} \forall_{\lambda \in [0,1]} \lambda x + (1-\lambda) y \in \mathrm{C}$*

Definition 56 (Convex combination). *Let $\{x_i\}_{i=1}^m \subseteq \mathbb{R}^n, \{\lambda_i\}_{i=1}^m \subseteq \mathbb{R}_+$, $\sum \lambda_i = 1$. The vector $\sum \lambda_i x_i = 1$ is called a convex combination of $\{x_i\}$.*

Lemma 23. C *is convex* $\iff$ C *contains all convex combinations of its elements.*

Definition 57 (Hyperplane). $H \subseteq \mathbb{R}^n$ *is hyperplane* $\iff \exists_{\beta \in \mathbb{R}, b \in \mathbb{R}^n} H = \{x \in \mathbb{R}^n : x \cdot b = \beta\}$

Lemma 24 (Hyperplane generates two halfspaces). $\{x \in \mathbb{R}^n : x \cdot b \leq \beta\}$ *and* $\{x \in \mathbb{R}^n : x \cdot b \geq \beta\}$

Definition 58 (Convex hull). . *Let* $Co = \cap\{C : E \subseteq C, C \text{ convex }\}$.
Note $Co = \left\{x \in \mathbb{R}^n : \exists_{\{x_i\} \subseteq E} \exists_{\{\lambda_i\} \subseteq \mathbb{R} : \sum \lambda_i = 1} x = \sum \lambda_i x_i\right\}$

Definition 59 (Simplex). . *A set* $S \subseteq \mathbb{R}^n$ *is* m *-dimensional simplex* $\iff S = \{(b_0, \ldots, b_m) \in \mathbb{R}^m : b_i \text{ affinely independent }\}$

Theorem 24. *If* $\forall_i C_i$ *convex following sets are convex*

- $\mathrm{C} = \bigcap C_i$
- $C_1 + a$
- $C_1 + C_2$ *is convex*
- $\mathrm{C} = \{x \in \mathbb{R}^n : x \cdot b \leq \beta\}$
- *if* f *is quasi concave function, then* $\mathrm{C} = \{x \in \mathbb{R}^n : f(x) \leq \beta\}$ *is convex*

Theorem 25 (Separating hyperplane theorem). . *Let* $C_1 \subseteq \mathbb{R}^n, C_2 \subseteq \mathbb{R}^n$. $H = \{x \in \mathbb{R}^n : x \cdot b = \beta\}$ *is separating hyperplane of* $C_1 \& C_2 \iff$

$$\forall_{x \in C_1} x \cdot b \leq \beta \qquad \forall_{y \in C_2} y \cdot b \geq \beta$$

Separation is strong if at least one is $<$, $>$)
" $\iff$ " part of theorem is true when

- $ri\,(C_1) \cap ri\,(C_2) = \emptyset$
- $ri(A) = \{x \in A : B(x, \epsilon) \cap aff(A) \subseteq A\}$
- $aff(A) = \{\sum \alpha_i x_i : x_i \in A, \sum \alpha_i = 1\}$

Conditions for "$\Rightarrow$ " separating hyperplane theorem: for all C_1, C_2 non empty, convex, $x \in \mathbb{R}^n$

- $x \notin C_1 \Rightarrow H(b, \beta)$ separates strongly $x \& C_1$
- $C_1 \cap C_2 = \emptyset \Rightarrow H(b, \beta)$ separates $C_1 \& C_2$
- C_1 open $\Rightarrow H(b, \beta)$ separates strongly $C_1 \& C_2$
- C_1, C_2 closed, C_1 compact$\Rightarrow H(b, \beta)$ separates strongly $C_1 \& C_2$

Definition 60. ***(Support)***. *The support function* $S(\cdot \mid C)$ *of convex set* $C \subseteq \mathbb{R}^n$ *is defined as:*

$$S(x, y) = \sup_{y \in \mathrm{C}} x \cdot y$$

Definition 61. ***(Concave function)*** *f is concave if for all $t \in [0,1]$ and for all x and y in C,*

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

Lemma 25. *f is concave if and only if the set*

$$\{(x, \alpha) \in C \times \mathbb{R} : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^{n+1}$. The set above is called ***hypograph*** *of f*

Lemma 26. ***(Jensen's Inequality)*** *f is concave if and only if $f(\lambda_1 x_1 + \ldots + \lambda_k x_k) \geq \lambda_1 f(x_1) + \ldots + \lambda_k f(x_k)$ for $x_1, \ldots, x_k \in \Gamma$ and $\lambda_i \geq 0$ and $\sum \lambda_1 = 1$*

Lemma 27. *C is open and f is differentiable on $C.f$ is concave if and only if for all x and y in C,*

$$f(x) \leq f(y) + \nabla f(y) \cdot (x - y)$$

Lemma 28. C *is open and f is twice continuously differentiableon C . f is concave if and only if for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T \leq 0, \forall v \in \mathbb{R}^n$$

Definition 62 (Strictly concave function). *f is strictly concave if for all $t \in]0,1[$ and for all x and y in C with $x \neq y$*

$$f(tx + (1-t)y) > tf(x) + (1-t)f(y)$$

Lemma 29. *C is open and f is differentiable on $C.f$ is strictly concave if and only if for all x and y in C with $x \neq y$,*

$$f(x) < f(y) + \nabla f(y) \cdot (x - y)$$

Lemma 30. *C is open and f is twice continuously differentiable on C. If for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T < 0, \forall v \in \mathbb{R}^n, v \neq 0$$

then f is strictly concave

Lemma 31. ***Monotone transformation*** *If f quasi convex, g monotone, nondecreasing, then $g \circ f$ is quasi-convex.*

Definition 63 (Quasi-concave function). *f is quasi-concave if and only if for all $\alpha \in \mathbb{R}$ the set*

$$\{x \in C : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^n$. The set above is called upper contour set of f at α.

Lemma 32. *f is quasi-concave if and only if for all $t \in [0,1]$ and for all x and y in C,*

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$$

Lemma 33. *C is open and f is differentiable on $C.f$ is quasiconcave if and only if for all x and y in C,*

$$f(x) \geq f(y) \Longrightarrow \nabla f(y) \cdot (x - y) \geq 0$$

Lemma 34. *C is open and f is differentiable on C. If f is quasiconcave and $\nabla f(x) \neq 0$ for all $x \in C$, then for all x and y in C with $x \neq y$,*

$$f(x) > f(y) \Longrightarrow \nabla f(y) \cdot (x - y) > 0$$

Definition 64. ***(Kernel)*** $Ker_{g(x)} := \{v \in \mathbb{R}^n, v \neq 0 \text{ and } \nabla g(x) \cdot v = 0\}$

Lemma 35. *C is open and f is twice continuously differentiable on C. If f is quasi-concave, then for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite on $Ker\nabla f(x)$, that is, for all $x \in C$*

$$v \in \mathbb{R}^n \quad \text{and} \quad \nabla f(x) \cdot v = 0 \Longrightarrow v\mathrm{H}f(x)v^T \leq 0$$

Definition 65. ***(Strictly quasi-concave function)*** *f is strictly quasi-concave if and only if for all $t \in]0, 1[$ and for all x and y in C with $x \neq y$,*

$$f(tx + (1 - t)y) > \min\{f(x), f(y)\}$$

Is concave function differentiable? almost everywhere. Moreover derivative is continuous a.s.

Lemma 36. *If f concave, $|f(x)| \leq M$ on open neighborhood of convex X, then f continuous.*

Lemma 37. *C is open and f is differentiable on C.*

1. *If for all x and y in C with $x \neq y$,*

   $$f(x) \geq f(y) \Longrightarrow \nabla f(y) \cdot (x - y) > 0$$

   then f is strictly quasi-concave

2. *If f is strictly quasi-concave and $\nabla f(x) \neq 0$ for all $x \in C$, then for all $x, y \in C$, $x \neq y$*

   $$f(x) \geq f(y) \Longrightarrow \nabla f(y) \cdot (x - y) > 0$$

Lemma 38. *C is open and f is twice continuously differentiable on C. If for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite on $Ker\nabla f(x)$, that is, for all $x \in C$*

$$v \in Ker\nabla f(x), \quad \Longrightarrow v\mathrm{H}f(x)v^T < 0$$

then f is strictly quasi-concave

Lemma 39. *We remark that*

$$\begin{array}{ccccc} f \text{ linear or affine} & \Rightarrow & f \text{ concave} & \Leftarrow & f \text{ strictly concave} \\ & & \downarrow & & \\ & & f \text{ quasi-concave} & \Leftarrow & f \text{ strictly quasi-concave} \end{array}$$

We remind the definitions and some properties of negative definite/semidefinite matrices. Let H be a $n \times n$ symmetric matrix.

Definition 66. ***(nsd, nd matrix)***

- H *is* ***negative semidefinite (nsd)*** *if* vHv$^T \le 0$ *for all* $v \in \mathbb{R}^n$
- *H is* ***negative definite (nd)*** *if* vHv$^T < 0$ *for all* $v \in \mathbb{R}^n$ *with* $v \neq 0$

Theorem 26. ***(Eigen values and definitness)***

1. *H has n real eigenvalues. We denote* $\lambda_1, \ldots, \lambda_n$ *the eigenvalues of H.*
2. *H is negative semidefinite if and only* $\lambda_i \le 0$ *for every* $i = 1, ..., n$
3. *H is negative definite if and only* $\lambda_i < 0$ *for every* $i = 1, ..., n$

Theorem 27. *(*$n = 2$ ***and definitness of matrix)***

1. *If H is negative semidefinite, then tr(H)* ≤ 0 *and det(H)* ≥ 0 *if n is even, det(H)* ≤ 0 *if n is odd*
2. *If H is negative definite, then tr(H)* < 0 *and det(H)* > 0 *if n is even, det(H)* < 0 *if n is odd*

We remark that if n = 2, then the conditions stated in the proposition above also are sufficient conditions, that is

1. *H is negative semidefinite if and only if tr(H)* ≤ 0 *and det(H)* ≥ 0*.*
2. *H is negative definite if and only if tr(H)* < 0 *and det(H)* > 0*.*

4 Optimization

4.1 Karush-Kuhn-Tucker Conditions

In this section, we assume that $C \subseteq \mathbb{R}^n$ is convex and open
- the following functions f and g_j with $j = 1, \ldots, m$ are differentiable on C

$$f : x \in C \subseteq \mathbb{R}^n \longrightarrow f(x) \in \mathbb{R} \text{ and}$$
$$g_j : x \in C \subseteq \mathbb{R}^n \longrightarrow g_j(x) \in \mathbb{R}, \forall j = 1, \ldots, m$$

Maximization problem

$$\max_{x \in C} f(x)$$
$$\text{subject to } g_j(x) \ge 0, \forall j = 1, \ldots, m$$

where f is the objective function, and g_j with $j = 1, \ldots, m$ are the constraint functions.

The Karush-Kuhn-Tucker conditions associated with problem are given below

$$\begin{cases} \nabla f(x) + \sum_{j=1}^m \lambda_j \nabla g_j(x) = 0 \\ \lambda_j g_j(x) = 0, \forall j = 1, \ldots, m \\ g_j(x) \ge 0, \forall j = 1, \ldots, m \\ \lambda_j \ge 0, \forall j = 1, \ldots, m \end{cases}$$

where for every $j = 1, \ldots, m, \lambda_j \in \mathbb{R}$ is called Lagrange multiplier associated with the inequality constraint g_j

Definition 67. *Let $x^* \in C$, we say that the constraint j is binding at x^* if f $g_j(x^*) = 0$. We denote*

1. *$B(x^*)$ the set of all binding constraints at x^*, that is*

$$B(x^*) := \{j = 1, \ldots, m : g_j(x^*) = 0\}$$

2. *$m^* \leq m$ the number of elements of $B(x^*)$ and*

3. *$g^* := (g_j)_{j \in B(x^*)}$ the following mapping*

$$g^* : x \in C \subseteq \mathbb{R}^n \longrightarrow g^*(x) = (g_j(x))_{j \in B(x^*)} \in \mathbb{R}^{m^*}$$

Theorem 28 (Karush-Kuhn-Tucker necessary conditions). *Let x^* be a solution to problem above. Assume that one of the following conditions is satisfied.*

1. *For all $j = 1, \ldots, m, g_j$ is a **linear** or **affine function**.*

2. *Slater's Condition :*

*for all $j = 1, \ldots, m, g_j$ is a **concave** function or g_j is a **quasiconcave** function with $\nabla g_j(x) \neq 0$ for all $x \in C$, and there exists $y \in C$ such that $g_j(y) > 0$ for all $j = 1, \ldots, m$*

3. ***Rank Condition** : rank $\mathrm{J}g^*(x^*) = m^* \leq n$ Then, there exists $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{R}_+^m$ such that (x^*, λ^*) satisfies the Karush-Kuhn-Tucker Conditions .*

Theorem 29 (Karush-Kuhn-Tucker sufficient conditions). *Suppose that there exists $\lambda^* = \left(\lambda_1^*, \ldots, \lambda_j^*, \ldots, \lambda_m^*\right) \in \mathbb{R}_+^m$ such that $(x^*, \lambda^*) \in C \times \mathbb{R}_+^m$*

satisfies the Karush-Kuhn-Tucker Conditions (2). Assume that

1. *f is a **concave** function or f is a **quasi-concave** function with $\nabla f(x) \neq 0$ for all $x \in C$, and*

2. *g_j is a **quasi-concave** function for all $j = 1, \ldots, m$*

Then, x^* is a solution to problem.

5 Correspondences

Let $\Theta \subseteq \mathbb{R}^n, X \subseteq \mathbb{R}^n$.

Definition 68. *A correspondence $\Gamma : \Theta \rightrightarrows X$ is a map s.t. $\Gamma(\Theta) \subseteq X$. $\left(\Gamma : \Theta \to 2^X\right)$*

Definition 69 (Graph of correspondence). . *$Gr(\Gamma) = \{(\theta, x) : \theta \in \Theta, x \in \Gamma(\theta)\}$*

Definition 70. ***(Properties of correspondences)**.*

1. ***not empty valued** if $\Gamma(\theta) \neq \emptyset \quad \forall \theta$*

2. ***single valued*** *if* $|\Gamma(\theta)| = 1 \quad \forall\theta$

3. ***closed valued*** *if* $\Gamma(\theta)$ *is closed set* $\forall\theta$

4. ***compact valued*** *if* $\Gamma(\theta)$ *is compact set* $\forall\theta$

5. ***convex valued*** *if* $\Gamma(\theta)$ *is convex set* $\forall\theta$

6. ***closed (graph)*** *if* $Gr(\Gamma)$ *is closed subset of* $\mathbb{E} \times X$

7. ***convex (graph)*** *if* $Gr(\Gamma)$ *is convex on* $\Theta \times X$

Lemma 40. *$Gr(\Gamma)$ is closed graph* $\iff \forall_{\theta:\theta_n\to\theta}\forall_{x_n\to x} : x_n \in \Gamma(\theta_n) \Rightarrow \quad x \in \Gamma(\theta)$

Lemma 41. *$Gr(\Gamma)$ is convex graph* $\iff \forall_\theta, \theta', x \in \Gamma(\theta), x' \in \Gamma(\theta')$ *it holds that* $\lambda x + (1-\lambda)x' \in \Gamma(\theta\lambda + (1-\lambda)\theta') \, \forall_{x\in[0,1]}$

Lemma 42. *$\Gamma : \Theta \rightrightarrows X$ has closed graph $\Rightarrow$ it is closed valued. If X is compact, than Γ is also compact valued.*

Definition 71 (Upper Hemi-Continuity). *Let $\Gamma : \Theta \rightrightarrows X$ be a correspondence.*

- *Γ is said to be* ***upper hemi-continuous (uhc)*** *at a point $\theta \in \Theta$ if and only if for all open sets $V \subseteq X$ such that $\Gamma(\theta) \subseteq V$, there exists an open set $U \subseteq \Theta$ such that $\theta \in U$ and for all $\theta' \in U$ it holds that $\Gamma(\theta') \subseteq V$*

- *A compact valued correspondence $\Gamma : \Theta \rightrightarrows X$ is u.h.c. at $\theta \in \Theta$ if and only if for every $\{\theta_n\} \subset \Theta$ such that $\theta_n \to \theta$ and every sequence $\{x_n\} \subset X$ such that $x_n \in \Gamma(\theta_n)$ there exits a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in \Gamma(\theta)$*
$$\forall_{\theta_n\to\theta}\forall_{x_n\in\Gamma(\theta_n)}\exists_{\{x_{n_k}\}} x_{n_k} \to x \in \Gamma(\theta)$$

Definition 72 (Lower Hemi-Continuity). . *Let $\Gamma : \Theta \rightrightarrows X$ be a correspondence.*

- *Γ is said to be* ***lower hemi-continuous (1hc)*** *at a point $\theta \in \Theta$ if and only if for all open sets $V \subseteq X$ such that $\Gamma(\theta) \cap V \neq \emptyset$, there exists an open set $U \subseteq \Theta$ such that $\theta \in U$ and for all $\theta' \in U$ it holds that $\Gamma(\theta') \cap V \neq \emptyset$*

- *A correspondence $\Gamma : \Theta \rightrightarrows X$ is l.h.c. at $\theta \in \Theta$ if for all $x \in \Gamma(\theta)$ and all sequences $\{\theta_n\} \subseteq \theta$ such that $\theta_n \to \theta$ there exits a sequence $\{x_n\} \subseteq X$ such that $x_n \in \Gamma(\theta_n)$ and $x_n \to x$*

$$\forall_{\theta_n\to\theta}\forall_{x\in\Gamma(\theta)}\exists_{x_n\in\Gamma(\theta_n)} x_n \to x$$

Definition 73 (Continuity). *Γ is said to be continuous at a point $\theta \in \Theta$ if it is both UHC an LHC.*

Lemma 43 (u.h.c and Closed graph). *Let $\Gamma : \Theta \rightrightarrows X$. If Γ is u.h.c, then Γ is closed (has a closed graph).*

Lemma 44 (Closed graph and u.h.c.). *Let $\Gamma : \Theta \rightrightarrows X$. If X is compact and Γ is closed (has a closed graph), then Γ is u.h.c.*

Theorem 30 (Berge (1961) of Maximum). ***Let*** $\Theta \subseteq \mathbb{R}^m$ ***and*** $X \subseteq \mathbb{R}^n$, ***let*** $f : \Theta \times X \to \mathbb{R}$ ***be a continuous function and*** $\Gamma : \Theta \rightrightarrows X$ ***a nonempty, compact valued, continuous correspondence. Define:***

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

Then

- $v : \Theta \to X$ ***is continuous***
- $G : \Theta \rightrightarrows X$ ***is nonempty and compact valued, and UHC***

Proof. The proof is divided in three parts. First it is proven that G *is nonempty and compact valued, then that it is u.h.c. and finally that* v *is continuous.*

1. G *is nonempty valued and compact valued.*
   - *Let* $\theta \in \Theta$*, by hypothesis* $\Gamma(\theta)$ *is compact and nonempty. since* $f(\cdot, \theta)$ *is continuous a maximum is attained on* $\Gamma(\theta)$ *by the extreme value theorem (Weierstrass). This proves that* $G(\theta)$ *is nonempty for arbitrary* θ*.*
   - *Let* $\theta \in \Theta$*, by hypothesis* $\Gamma(\theta)$ *is compact and nonempty. since* $G(\theta) \subseteq \Gamma(\theta)$ *it follows that* $G(\theta)$ *is bounded, it is left to show closedness to establish compactness. Let* $x_n \to x$ *and* $x_n \in G(\theta)$ *for all* n*. Clearly* $x_n \in \Gamma(\theta)$ *for all* n*, since* Γ *is closed valued it follows that* $x \in \Gamma(\theta)$*, so its feasible. By definition of* G *we have* $v(\theta) = f(x_n, \theta)$ *for all* n*, since* f *is continuous we get* $v(\theta) = \lim f(x_n, \theta) = f(x, \theta)$*, then by definition* $x \in G(\theta)$*, which proves closedness.*
2. G *is u.h.c. Consider* $\theta \in \Theta$*, a sequence in* Θ *such that* $\theta_n \to \theta$ *and a sequence in* X *such that* $x_n \in G(\theta_n)$ *for all* n*. Note that* $x_n \in \Gamma(\theta_n)$*. since* Γ *is u.h.c. there exists a subsequence* $x_{n_k} \to x \in \Gamma(\theta)$ *Now consider* $z \in \Gamma(\theta)$*. since* Γ *is l.h.c. there exists a sequence in* X *such that* $z_n \in \Gamma(\theta_n)$ *and* $z_n \to z$*. In particular the subsequence* $\{z_{n_k}\}$ *also converges to* z *since* $x_n \in G(\theta_n)$ *and* $z_n \in \Gamma(\theta_n)$ *it follows that* $f(x_n, \theta_n) \geq f(z_n, \theta_n)$*. since* f *is continuous in both arguments we get by taking limits:* $f(x, \theta) \geq f(z, \theta)$*. since the inequality holds for arbitrary* $z \in \Gamma(\theta)$ *we get the result:* $x \in G(\theta)$*. This proves u.h.c.*
3. v *is continuous. Let* $\theta \in \Theta$ *and* $\theta_n \to \theta$ *an arbitrary sequence converging to* θ*. Consider an arbitrary sequence in* X *such that* $x_n \in G(\theta_n)$ *for all* n*. Let* $\bar{v} = \limsup v(\theta_n)$*. By proposition 2.9 there is a subsequence* $\{\theta_{n_k}\}$ *such that* $v(\theta_{n_k}) \to \bar{v}$*. since* G *is u.h.c. there exists a subsequence of* $\{x_{n_k}\}$ *(call it* $\{x_{n_{k_l}}\}$*) converging to a point* $x \in G(\theta)$*. Then*

   $$\bar{v} = \lim v(\theta_{k_l}) = \lim f(x_{k_l}, \theta_{k_l}) = f(x, \theta) = v(\theta)$$

   where the second equality follows from $x_{k_l} \in G(\theta_{k_l})$*, the third one from* f *being continuous and the final one from* $x \in G(\theta)$*. Let* $\underline{v} = \liminf v(\theta_n)$ *and by a similar argument we get* $v(\theta) = \underline{v}$ *since* $v(\theta) = \liminf v(\theta_n) = \limsup v(\theta_n)$ *we get* $v(\theta) = \lim v(\theta_n)$ *for arbitrary* $\{\theta_n\}$ *converging to* θ*. This proves continuity.*

Theorem 31 (ToM under convexity). *Let* $\Theta \subseteq \mathbb{R}^m$ *and* $X \subseteq \mathbb{R}^n$, *let* $f : \Theta \times X \to \mathbb{R}$ *be a continuous function and* $\Gamma : \Theta \Rightarrow X$ *a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

a If $f(\cdot, \theta)$ *is concave in* x *for all* θ *and* Γ *is convex valued then* G *is convex valued.*

b If $f(\cdot, \theta)$ *is strictly concave in* x *for all* θ *and* Γ *is convex valued then* G *is single valued, hence a continuous function.*

c If f *is concave on* $\Theta \times X$ *and* Γ *has a convex graph then* v *is concave and* G *is convex valued.*

d If f *is strictly concave on* $\Theta \times X$ *and* Γ *has a convex graph then* v *is strictly concave and* G *is single valued, hence a continuous function.*

Theorem 32 (ToM under quasi-convexity). . *Let* $\Theta \subseteq \mathbb{R}^m$ *and* $X \subseteq \mathbb{R}^n$, *let* $f : \Theta \times X \to \mathbb{R}$ *be a continuous function and* $\Gamma : \Theta \rightrightarrows X$ *a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

a If $f(\cdot, \theta)$ *is quasi-concave in* x *for all* θ *and* Γ *is convex valued then* G *is convex valued.*

b If $f(\cdot, \theta)$ *is strictly quasi-concave in* x *for all* θ *and* Γ *is convex valued then* G *is single valued, hence a continuous function.*

c If f *is quasi-concave on* $\Theta \times X$ *and* Γ *has a convex graph then* v *is quasi-concave and* G *is quasi-convex valued.*

d If f *is strictly quasi-concave on* $\Theta \times X$ *and* Γ *has a convex graph then* v *is strictly quasi-concave and* G *is single valued, hence a continuous function.*

5.1 Berge theorem applied to micro

One, useful, application of the material covered above is to determine properties of the budget correspondence, that indicates the feasible consumption bundles for a consumer given a price vector p and an endowment vector e. Suppose there are l goods, and that the agent has a fixed endowment of each good given by the vector $e \in \mathbb{R}^l_{++}$, the price of the goods is a vector $p \in \Delta$, where Δ is the n -dimensional open simplex. Define the budget set correspondence $B(\cdot, e) : \Delta \rightrightarrows \mathbb{R}^l_+$by

$$B(p, e) = \left\{x \in \mathbb{R}^l_+ \mid p \cdot x \leq p \cdot e\right\}$$

Theorem 33. $B(\cdot, e)$ *is continuous on prices.*

Proof. The claim is proved establishing u.h.c. and l.h.c. of B.

1. $B(\cdot, e)$ is upper hemi-continuous on prices. Let $p \in \Delta, \{p_n\} \subseteq \Delta$ with $p_n \to p$ and $\{x_n\} \subseteq \mathbb{R}^l_+$ a sequence such that $x_n \in B(p_n, e)$ since $p_n \to p \in \Delta$ there exists a closed ball, C, around p such that $C \subseteq \Delta$ and for n large enough $p_n \in C$. Let $\xi_i = \max_{p \in C} \frac{p \cdot e}{p_i}$ for $i = 1, \ldots, l$. ξ_i is the maximum amount of x_i that can be bought in the neighborhood of p. Define $\xi = \max\{\xi_i\} + 1$, it is clear that for n large enough $x_n \in B_\xi(0)$, then $\{x_n\}$ is a bounded sequence, hence it admits a convergent subsequence $x_{n_k} \to x$. since $x_{n_k} \in B(p_{n_k}, e)$ we have: $p_{n_k} \cdot x_{n_k} \leq p_{n_k} \cdot e$, since dot product is a continuous function taking limits we have $p \cdot x \leq p \cdot e$, which is $x \in B(p, e)$, proving u.h.c. of B.

2. $B(\cdot, e)$ is lower hemi-continuous on prices. Let $p \in \Delta, \{p_n\} \subseteq \Delta$ with $p_n \to p$ and $x \in B(p, e)$. Define $\eta^i_n = \max\left\{0, \frac{p_n \cdot x - p_n \cdot e}{l p^i_n}\right\}$ and let $x_n = x - \eta_n$ Clearly $x_n \in B(p_n, e)$ since either $x \in B(p_n, e)$ or

$$p_n \cdot x_n = p_n \cdot x - \sum p^i_n \left(\frac{p_n \cdot x - p_n \cdot e}{l p^i_n}\right) = p_n \cdot x - (p_n \cdot x - p_n \cdot e) = p_n \cdot e$$

then $p_n \cdot x_n \leq p_n \cdot e$ Moreover $x_n \to x$, since $x \in B(p, e)$ and $p_n \to p$ it follows that $p_n \cdot x - p_n \cdot e \to p \cdot x - p \cdot e \leq 0$, then $\eta_n = \max\{0, p_n \cdot x - p_n \cdot e\} \to 0$ which is $x_n \to x$. Then B is l.h.c.

3. Note that it wasn't checked if $x_n \geq 0$ for all n. This is not guaranteed by the construction above. With extra notation it can be guaranteed that $x^i_n \geq 0$.

The consumer problem is often laid out without explicit endowments of the goods, instead the parameters are prices $p \in \mathbb{R}^l_{++}$ and a nominal income level $I \in \mathbb{R}_+$. The set of parameters is $\Theta = \mathbb{R}^l_{++} \times \mathbb{R}$. The **indirect utility function** and the **Marshalian demand correspondence** are:

$$v(p, I) = \max_{x \in B(p,I)} u(x) \quad G(p, I) = \{x \in B(p, I) \mid u(x) = v(p, I)\}$$

where the budget set is given by the correspondence:

$$B(p, I) = \left\{x \in \mathbb{R}^l_+ \mid p \cdot x \leq I\right\}$$

I take as given that B is a nonempty, convex valued and continuous correspondence, and that u is a continuous function.

Theorem 34. *v and G have the following properties on Θ.*

a *v is a continuous function on Θ and G is a nonempty, compact valued, u.h.c. correspondence.*

b *v is nondecreasing in I for fixed p and non-increasing in p for fixed I.*

c *v is jointly quasi-convex on (p, I).*

d *If u is (quasi) concave then v is (quasi) concave in I for fixed p.*

e *If u is (quasi) concave then G is a convex valued correspondence.*

f *If u is strictly (quasi) concave then G is a continuous function.*

5.2 Nash equilibrium in normal form games

Definition 74. *A normal form game is formed by:*

a *A finite set of agents $I = \{1, \dots, N\}$. A generic player is denoted i and the set of other players $-i$.*

b *For each player a finite action set A_i. Note $A = \times A_i$.*

c *For each player a payoff function $u^i : A \to \mathbb{R}$.*

From the set of pure strategies of a player one can define the set of mixed strategies. $S_i = \Delta(A_i)$, a mixed strategy is a probability distribution over the set of possible actions A_i. Formally:

$$S_i = \Delta(A_i) = \left\{ s_i : A_i \to [0,1] \mid \sum_{a_i \in A_i} s_i(a_i) = 1 \right\}$$

Note that S_i is convex and compact. In fact S_i is the convex hull of A_i. If players play mixed strategies they rank alternative strategies according to their expected payoffs, the expected payoffs are given by function $v^i : S_i \times S_{-i} \to \mathbb{R}$ which is:

$$v^i(t, s_{-i}) = \sum_{a_i \in A_i} t(a_i) \left(\sum_{a_{-i} \in A_{-i}} \prod_{j \neq i} s_j(a_j)\, u^i(a_i, a_{-i}) \right) = \sum_{a \in A} \left(\left(t(a_i) \prod_{j \neq i} s_j(a_j) \right) u^i(a) \right)$$

In a game where players play simultaneously in a noncooperative manner they have to answer optimally to a given strategy profile of the other players.

Definition 75 (The best response). *of a player to s_{-i} is given by:*

$$\mathrm{BR}_i(s_i, s_{-i}) = \mathrm{BR}_i(s) = \left\{ t \in S_i \mid \forall_{r \in S_i} u^i(t, s_{-i}) \geq u^i(r, s_{-i}) \right\} = t \in S_i argmax v^i(t, s_{-i})$$

Note that BR^i is the solution to the problem $V(s) = \max_{t \in S_i} v^i(t, s_{-i})$ since S_i is a fixed set it is also a constant correspondence with argument s, a strategy profile. It is then continuous as well as nonempty, compact and convex valued. Moreover v^i is continuous in s_{-i} and constant in s_i by construction, then it is continuous in $s.v$ is also linear in t holding s_{-i} constant, then it is concave. It follows that the ToM under convexity applies, then the BR is a nonempty, compact and convex valued and u.h.c. correspondence for each player.

Definition 76 (A Nash Equilibrium). *is defined as a strategy profile $s^\star \in S$ such that $s_i^\star \in \mathrm{BR}_i(s^\star)$ for all i. A way to think about it is to form a correspondence with the cartesian product of the individual BR correspondences, this is* $\mathrm{BR} : S \to S$ *defined as:*

$$\mathrm{BR}(s) = \times \mathrm{BR}_i(s)$$

Note that BR is by construction a nonempty, compact and convex valued and u.h.c. correspondence.

A NE is then a fixed point of the correspondence BR. The following theorem will establish the existence of such fixed point.

6 Fixed Point Theorems

Theorem 35. ***Brouwer's Fixed Point Theorem – continuous function***
Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and $f : S \to S$ be a continuous function. Then f has (at least) a fixed point in S, i.e. $\exists x^* \in S : x^* = f(x^*)$

Theorem 36 (Tarsky's Fixed Point Theorem weakly increasing functions). *Let $f : [0,1]^n \to [0,1]^n$, where $[0,1]^n = [0,1] \times ... \times [0,1]$, an n-dimensional cube. If f is nondecreasing, then f has a fixed point in $[0,1]^n$.*

Theorem 37 (Kakutani's Fixed Point Theorem u.h.c. correspondence). *Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence. Then Γ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$*

Since S is compact, u.h.c. is equivalent to Γ having a closed graph.

Theorem 38. ***Fixed Point Theorem – l.h.c. correspondence***
Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, closed-valued, and l.h.c. correspondence. Then Γ has a fixed point in S.

7 Comparative statics ala Topkis

Definition 77 (Meet and Joint). *Given $x, y \in \mathbb{R}^n$, **the meet** of x and y, denoted $x \wedge y$, is*

$$x \wedge y = (\min\{x_1, y_1\}, \cdots, \min\{x_n, y_n\})$$

***The joint** of x and y, denoted $x \vee y$, is*

$$x \vee y = (\max\{x_1, y_2\}, \cdots, \max\{x_n, y_n\})$$

Definition 78 (Lattice). *$X \subseteq \mathbb{R}^n$ is a lattice of $\mathbb{R}^n$ if $\forall x, y \in X, x \wedge y \in X$ and $x \vee y \in X$*

Remark: A budget set is generally not a lattice of $\mathbb{R}^n$. More for lattice: we can define compact, sup/inf on it:

Definition 79 (compact lattice). *$X \subseteq \mathbb{R}^n$ is a compact lattice if X is a lattice and X is compact under the Euclidean metric.*

Definition 80. *$x^* \in X$ is a greatest element of lattice X if $x^* \geq x, \forall x \in X, \hat{x} \in X$ is a least element of lattice X if $\hat{x} \leq x, \forall x \in X$*

Definition 81. ***(Uniqueness of greatest and least element)** Suppose $X \subseteq \mathbb{R}^n$ is a non-empty, compact lattice. Then, X has a greatest element and a least element.*

Definition 82 (Supermodular). *$f : S \times \Theta \to \mathbb{R}$ is supermodular in (x, θ) if $\forall z = (x, \theta)$ and*

$$z' = (x', \theta') \, in \quad S \times \Theta f(z) + f(z') \leq f(z \vee z') + f(z \wedge z')$$

Theorem 39 (Supermodularity). *$f : S \times \Theta \to \mathbb{R}$ is supermodular in (x, θ), then for any fixed θ, f is supermodular in$x$, i.e.*

$$f(x, \theta) + f(x', \theta) \leq f(x \vee x', \theta) + f(x \wedge x', \theta))$$

Definition 83 (Increasing Differences). *$f : S \times \Theta \to \mathbb{R}$ satisfies increasing differences in (x, θ) if $\forall (x, \theta), (x', \theta') \in S \times \Theta$ such that $x \geq x'$ and $\theta \geq \theta'$*

$$f(x, \theta) - f(x', \theta) \geq f(x, \theta') - f(x', \theta')$$

If the inequality is strict whenever $x > x'$ and $\theta < \theta'$, then f satisfies strictly increasing differences $in(x, \theta)$

Theorem 40 (Supermodularity vs. Increasing Differences). *$f : Z \subseteq \mathbb{R}^n \to \mathbb{R}$ is supermodular in z iff f has increasing return in z*

Theorem 41 (Topkis' Characterization Theorem). *Let Z be an open lattice of $\mathbb{R}^n$. A $\mathcal{C}^2$ function $h : Z \to \mathbb{R}$ is supermodular on Z iff $\forall z \in Z$*

$$\frac{\partial^2 h}{\partial z_i \partial z_j}(z) \geq 0, \forall i \neq j$$

7.1 Parametric Monotonicity

Now let's consider the optimization problem:

$$\max_{x \in S} f(x; \theta)$$

with

$$f^*(\theta) = \max\{f(x; \theta) \mid x \in S\}, \quad D^*(\theta) = argmax\{f(x; \theta) \mid x \in S\}$$

A correspondence $D^*(\theta)$ is nondecreasing in θ if for every $\theta \leq \theta'$

$$D^*(\theta) \leq D^*(\theta')$$

Above inequality between sets means the strong set order: for every $x \in D^*(\theta)$ and $x' \in D^*(\theta')$, it holds $x \vee x' \in D^*(\theta'), x \wedge x' \in D^*(\theta)$

Theorem 42 (Topkis' Monotonicity Theorem). *Let S be compact lattice of $\mathbb{R}^n, \Theta$ be a lattice of $\mathbb{R}^l$, and $f : S \times \Theta \to \mathbb{R}$ be a continuous function on S, for each fixed θ. Suppose f satisfies increasing differences in (x, θ), and is supermodular in x for each fixed θ. Then D^* is nondecreasing in θ.*

8 Dynamic Programming

This part is based on 'Recursive Methods in Economic Dynamics' by Stokey, Lucas and Prescott (SLP in short) chapter 3-4.

8.1 Pointwise and Uniform Convergence

Definition 84. *(Pointwise convergence) Let $E \subseteq \mathbb{R}$ nonempty. A sequence of functions $f_n : E \to \mathbb{R}$ is said to converge pointwise on E if and only the limit* $\lim f_n(x) = f(x)$ *exists for all $x \in E$.*

Definition 85. *(Uniformly Convergence) Let $E \subseteq \mathbb{R}$ nonempty. A sequence of functions $f_n : E \to \mathbb{R}$ is said to converge uniformly on E if and only the*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall x \in E, \forall n \geq N, \quad |f_n(x) - f(x)| < \varepsilon$$

We write $f_n \rightrightarrows f$.

Definition 86 (Uniform norm). *Let $E \subseteq \mathbb{R}, \varphi : E \to \mathbb{R}$ is a function, we say φ is bounded on E if the set $\varphi(E)$ is a bounded subset of $\mathbb{R}$. If φ is bounded, we define the uniform norm of φ on E by*

$$\|\varphi\|_E := \sup\{|\varphi(x)| : x \in E\}$$

Note that it follows that $\varepsilon > 0$ then

$$\|\varphi\|_E \leq \varepsilon \quad \Longleftrightarrow \quad |\varphi(x)| \leq \varepsilon \quad \forall x \in E$$

Lemma 45. *A sequence $\{f_n\}$ of bounded functions on $E \subseteq \mathbb{R}$ converges uniformly on E to f if and only if $\|f_n - f\|_E \to 0$*

Lemma 46. *Suppose f is the limit of a bounded, uniformly convergence sequence $\{f_n\}$, then f is also bounded.*

Theorem 43. *(Cauchy Criterion for Uniform Convergence Let $\{f_n\}$ be a sequence of bounded functions on $E \subseteq \mathbb{R}$. Then this sequence converges uniformly on E to a bounded function f if and only if for every $\varepsilon > 0$ there is a number $N \in \mathbb{N}$ s.t. for all $m, n \geq N$, then*

$$\|f_m - f_n\|_E \leq \varepsilon$$

Theorem 44 (Interchange of Limit and Continuity). *Let $\{f_n\}$ be a sequence of continuous on a set $A \subseteq \mathbb{R}$ and suppose $\{f_n\}$ uniformly convergence to a function $f : A \to \mathbb{R}$. Then f is continuous on A*

Theorem 45 (Interchange of Limit and Derivative). *Let $E \subseteq \mathbb{R}$ be a bounded interval, $\{f_n\} : E \to \mathbb{R}$. Suppose (1) $\exists x_0 \in E$ s.t. $\{f_n(x_0)\}$ (pointwisely) converges 48 (2) $\{f_n'\}$ exists and converges uniformly on E to a function g Then $\{f_n\}$ converges uniformly on E to a function f that has a derivative at every point on E and $f' = g$. Or equivalently*

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'$$

Theorem 46 (Interchange of Limit and Integral). *Let $\{f_n\}$ be a sequence of function in $\mathcal{R}[a, b]$ and suppose $\{f_n\}$ converges uniformly on $[a, b]$ to f. Then $f \subset \mathcal{R}[a, b]$ and*

$$\int_a^b \lim_{n\to\infty} f_n = \int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

Definition 87 (Metric Space). *A metric space is a pair* (s, ρ) *of a set and a metric (distance function)* $\rho : S \times S \to \mathbb{R}$ *s.t. for all* $x, y, z \in S$

(1) $\rho(x, y) \geq 0$, *and* $\rho(x, y) = 0 \iff x = y$
(2) $\rho(x, y) = \rho(y, x)$
(3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Definition 88 (Contraction). *Let* (S, ρ) *be a metric space, and a function* $T : S \to S$ *mapping* S *into itself.* T *is a contraction (with modulus* β *) if and only if there exists* $\beta \in (0, 1)$ *s.t. for all* $x, y \in S$

$$\rho(Tx, Ty) \leq \beta \rho(x, y)$$

The iterates of T *are the mappings* $\{T^n\}$ *define by* $T^0(x) = x, T^n(x) = T\left(T^{n-1}(x)\right)$.

Definition 89 (Fixed point). $x \in S$ *is called a fixed point for* $T : S \to S$ *if* $Tx = x$.

Definition 90 (Space of bounded, continuous functions). *By* $(C(X), ||\cdot||_{\text{sup}})$ *we denote*

$$C(X) = \{f : X \to \mathbb{R}, \quad f \in \mathcal{C}^0, \quad ||f||_{\text{sup}} < \infty\}$$

with supremum norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$ *(with metric* $d(x, y) = ||x - y||_{\text{sup}}$*)*

1. By Lemma x pointwise convergence and uniformly convergence are equivalent for the bounded function under sup-norm. (Hence in this metric space, we don't need to worry about the types of convergence, and all the good properties about interchange follows.)

2. By Theorem y a sequence of bounded functions uniformly convergence to a bounded function if and only if it is a Cauchy sequence. This implies the space composed by bounded functions $B(X)$ together with sup-norm is complete. Here, we can also show the space related to $C(X)$ is also complete:

Theorem 47. $C(X)$ *with sup-norm is a complete normed vector space.*

Theorem 48 (Banach, Contraction Mapping Theorem). *If* (S, ρ) *is a complete metric space and* $T : S \to S$ *is a contraction mapping with modulus* β*, then*

1. T *has exactly one fixed point* v *in* S

2. *For any* $v_0 \in S, \rho\left(T^n v_0, v\right) \leq \beta^n \rho\left(v_0, v\right), n = 0, 1, 2, \cdots$

Lemma 47. *Let* (S, ρ) *be a complete metric space, and let* $T : S \to S$ *be a contraction mapping with fixed point* $v \in S$*. If* S' *is a closed subset of* S *and* $T\left(S'\right) \subset S'$*, then* $v \in S'$*. If in addition* $T\left(S'\right) \subset S'' \subset S'$*, then* $v \in S''$

Lemma 48. *(N-stage Contraction Theorem) Let* (S, ρ) *be a complete metric space, let* $T : S \to S$*, and suppose that for some integer* $N, T^N : S \to S$ *is a contraction mapping with modulus* β*. Then (a)* T *has exactly one fixed point on* S*, (b) for any* $v_0 \in S, \rho\left(T^{kN} v_0, v\right) \leq \beta^k \rho\left(v_0, v\right), k = 0, 1, 2, \cdots$

Theorem 49. *(Blackwell's sufficient conditions for a contraction) Let $X \subset \mathbb{R}^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying 1. (Monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x), \forall x \in X$, implies $(Tf)(x) \leq (Tg)(x), \forall x \in X$ 2. (Discounting) there exists some $\beta \in (0,1)$ such that*

$$[T(f+a)](x) \leq (Tf)(x) + \beta a, \forall f \in B(X), a \geq 0, x \in X$$

where $(f+a)$ is a function defined by $(f+a)(x) = f(x)+a$ Then T is a contraction with modulus β.

Lemma (3.7). *Let $X \subset \mathbb{R}^l$ and $Y \subset \mathbb{R}^n$. Assume that the correspondence $\Gamma : X \to Y$ is non-empty, compact- and convex-valued, and continuous, and let A be the graph of Γ. Assume that the function $f : A \to \mathbb{R}$ is continuous and that $f(x, \cdot)$ is strictly concave, for each $x \in X$. Define the function $g : X \to Y$ by*

$$g(x) = \arg \max_{y \in \Gamma(x)} f(x, y)$$

Then for each $\epsilon > 0$ and $x \in X$, there exists $\delta_x > 0$ such that

$$y \in \Gamma(x) and |f[x, g(x)] - f(x, y)| < \delta_x \Rightarrow \|g(x) - y\| < \epsilon$$

If X is compact, then $\delta > 0$ can be chosen independently of x.

Theorem (3.8). *Let X, Y, Γ and A be as defined in Lemma 3.7. Let $\{f_n\}$ be a sequence of continuous (real-valued) functions on A; assume that for each n and each $x \in X, f_n(x, \cdot)$ is strictly concave in its second argument. Assume that f has the same properties and that $f_n \to f$ uniformly (in the sup norm). Define the functions g_n and g by*

$$g_n(x) = \arg \max_{y \in \Gamma(x)} f_n(x, y), n = 1, 2, \cdots g(x) = \arg \max_{y \in \Gamma(x)} f(x, y)$$

Then $g_n \to g$ pointwise. If X is compact, $g_n \to g$ uniformly.

8.2 Certainty and Bounded returns

The problem to be studied in terms of infinite sequences is of the form (SP):

$$v^\star(x_0) = \sup_{\{x_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t F(x_t, x_{t+1}) \quad s.t. x_{t+1} \in \Gamma(x_t)$$

Corresponding to this problem is the following functional equation (FE):

$$v(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

Where

- X denotes the set of possible values for the state variable $x . X$ could be a subset of a Euclidean space, or any other set.
- $\Gamma : X \rightrightarrows X$ is the correspondence describing feasibility constraint.

- Let A be the graph of Γ, i.e.

$$A = \{(x, y) \in X \times X \mid y \in \Gamma(X)\}$$

where x denotes today's state and y denotes tomorrow's state.

- $F : A \to \mathbb{R}$ is the one-period return function.
- $\beta \geq 0$ is the (stationary) discount factor.
- Let x_0 be the initial state.

Definition 91. *Let's call any sequences $\{x_t\}_{t=0}^{\infty}$ in X a **plan**, given $x_0 \in X$, let denote $\Pi(x_0)$ as the set of **feasible plan** from x_0 :*

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} \mid x_{t+1} \in \Gamma(x_t), \forall t = 0, 1, \cdots\}$$

i.e., $\Pi(x_0)$ is the set of all sequences $\{x_t\}_{t=0}^{\infty}$ satisfying the constraints in (SP).

Definition 92. *Let $\vec{x} = (x_0, x_1, \cdots)$ denote a typical element of $\Pi(x_0)$, the utility of this plan is*

$$u(\vec{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

Definition 93. *Denote $v^*(x_0) = \sup_{\vec{x} \in \Pi(x_0)} u(\vec{x})$*

8.3 Assumptions

Assumption (4.1). *$\Gamma(x)$ is nonempty, for all $x \in X$*

Assumption (4.2). *$\forall x_0 \in X$ and $\vec{x} \in \Pi(x_0), \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists (although it may be plus or minus infinity).*

Assumption (4.3). *X is a convex subset of $\mathbf{R}^l$, and the correspondence $\Gamma : X \to X$ is nonempty, compact-valued and continuous.*

Assumption (4.4). *The function $F : A \to \mathbf{R}$ is bounded and continuous, and $0 < \beta < 1$*

Assumption (4.5). *For each $y, F(x, y)$ is a strictly increasing in each of its first l arguments.*

Assumption (4.6). *Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$*

Assumption (4.7). *F is strictly concave, that is $F(\theta(x, y) + (1-\theta)(x', y')) \geq \theta F(x, y) + (1-\theta) F(x', y')$, all $(x, y), (x'y') \in A$, and all $\theta \in (0, 1)$ and the inequality is strict is $x \neq x'$.*

Assumption (4.8). *Γ is convex in the sense that for any $\theta \in (0, 1)$, and any $x, x' \in Xy \in \Gamma(x)$ $y' \in \Gamma(x')$ implies $\theta y + (1-\theta) y' \in \Gamma(\theta x + (1-\theta) x')$*

Lemma (4.1). *Let X, Γ, F and β satisfy Assumption 4.2. Then for any $x_0 \in X$, and any $(x_0, x_1, \cdots) = \vec{x} \in \Pi(x_0)$*

$$u(x) = F(x_0, x_1) + \beta u(x') \quad where x' = (x_1, x_2, \cdots)$$

Assumption (4.9). *F is continuously differentiable on the interior of A.*

Theorem (4.2). *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. Then the function v^* satisfies (FE).*

Theorem (4.3). *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. If v is the solution to (FE) and satisfies*

$$\lim_{n\to\infty} \beta^n v\left(x_n\right) = 0, all\left(x_0, x_1, \ldots\right) \in \Pi\left(x_0\right), all x_0 \in X$$

then $v = v^*$.

Theorem (4.4). *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. Let $x^* \in \Pi\left(x_0\right)$ be a feasible plan that attains the supremum in (SP) for initial state x_0. Then*

$$v^*\left(x_t^*\right) = F\left(x_t^*, x_{t+1}^*\right) + \beta v\left(x_{t+1}^*\right), \quad t = 0, 1, 2, \ldots$$

Theorem (4.5). *Let X, Γ, F and β satisfy Assumptions 4.1-4.2. Let $x^* \in \Pi\left(x_0\right)$ be a feasible plan from x_0 satisfying*

$$v^*\left(x_t^*\right) = F\left(x_t^*, x_{t+1}^*\right) + \beta v\left(x_{t+1}^*\right), \quad t = 0, 1, 2, \ldots$$

and with

$$\limsup_{t\to\infty} \beta^t v^*\left(x_t^*\right) \leq 0$$

Then x^* attains the supremum in (SP) for initial state x_0.

Theorem (4.6). *Let X, Γ, F and β satisfy Assumptions 4.3-4.4, and let $C(X)$ be the space of bounded continuous functions $f : X \to \mathbf{R}$, with the supnorm $\|f\| = \sup_{x\in X}|f(x)|$. Then the operator T on $C(X)$ defined by*

$$(Tf)(x) = \max_{y\in\Gamma(x)}[F(x,y) + \beta f(y)]$$

maps $C(X)$ into itself, $T : C(X) \to C(X)$; it has a unique fixed point $v \in C(X)$; and for all $v_0 \in C(X)$

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = 0, 1, 2, \ldots$$

Moreover given v the optimal policy correspondence $G : X \to X$ defined by

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x,y) + \beta v(y)\}$$

is compact-valued and u.h.c.

Theorem (4.7). *Let X, Γ, F and β satisfy Assumptions 4.3-4.6, and let v be a unique solution to*

$$v(x) = \max_{y\in\Gamma(x)} F(x,y) + \beta v(y)$$

then v is strictly increasing

Theorem (4.8). *Let X, Γ, F and β satisfy Assumptions 4.3-4.4, and 4.7-4.8, let v be a unique solution to*

$$v(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y)$$

and G satisfy

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x,y) + \beta v(y)\}$$

then v is strictly concave and G is a continuous, single-valued function.

Theorem (4.11). *Let X, Γ, F and β satisfy Assumptions 4.3-4.4, and 4.7-4.9, let v and g satisfy*

$$v(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y)$$

and

$$g(x) = \{y \in \Gamma(x) : v(x) = F(x,y) + \beta v(y)\}$$

If $x_0 \in$ int X and $g(x_0) \in \Gamma(x_0)$, then v is continuously differentiable at x_0, with derivatives given by $v_i(x_0) = F_i(x_0, g(x_0)), i = 1, 2, \ldots, l$

Assumption (4.10). *$X \subseteq R^l$ is a convex cone. The correspondence $\Gamma : X \to X$ is nonempty, compact-valued, and continuous; for any $x \in X$,*

$$y \in \Gamma(x) implies \lambda y \in \Gamma(\lambda x), all \lambda > 0$$

that is, graph of Γ is cone. In addition, for some $\alpha \in (0, \beta^{-1})$,

$$\|y\|_l \leq \alpha \|x\|_l, all y \in \Gamma(x), all x \in X$$

Assumption (4.11). *The function $F : A \to \mathbf{R}$ is continuous and homogeneous of degree one, and for some $0 < B < \infty$*

$$|F(x,y)| \leq B\left(\|x\|_l + \|y\|_l\right), all (x,y) \in A$$

and $\beta \in (0,1)$

Theorem (4.12). *Let X be a convex cone, $H(X)$ be a space of continuous, homogeneous of degree 1 functions, bounded in the norm $\max_{||x||=1, x \in X} |f(x)|$, and let the mapping $T : H(X) \to H(X)$ satisfy*

1. *monotonicity: for any $f, g \in H(X)$, if $f \leq g$ then $Tf \leq Tg$*
2. *discounting: exists $\gamma \in (0,1)$ such that for all $f \in H$ and all $a > 0, T(f + a) \leq Tf + \gamma a$*

Then T is a contraction with modulus γ.

Theorem (4.13). *Let X, Γ, F and β satisfy Assumptions* 4.10 *and 4.11, and let $H(X)$ be a space of continuous, homogeneous of degree 1 functions, bounded in the norm $\max_{\|x\|=1, x \in X} |f(x)|$. Then operator T on $H(X)$ defined by*

$$(Tf)(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$$

maps $H(X)$ into itself, that is $T : H(X) \to H(X)$, and has a unique fixed point $v^* \in H(x)$. In addition

$$\|T^n v_0 - v^*\| \leq (\alpha\beta)^n \|v_0 - v^*\|, \quad n = 0, 1, 2, \ldots, all v_0 \in H(X)$$

and the associated policu correspondence $G : X \to X$ is a compact valued and u.h.c. Moreover, G is homogeneous of degree one, for any $x \in X$

$$y \in G(x) \text{ implies } \lambda y \in G(\lambda x), all \lambda \geq 0$$

Theorem (4.15). *(Sufficiency of the Euler and transversality conditions) Let $X \subset \mathbb{R}^l_+$, and let F satisfy Assumptions 4.3–4.5, 4.7 and 4.9. Then the sequence $\left\{x^*_{t+1}\right\}_{t=0}^{\infty}$, with $x^*_{t+1} \in \int \Gamma\left(x^*_t\right), t = 0, 1, \cdots$, is optimal for the problem (SP), given x_0, if it satisfies (2) and (3)*

9 Stochastic analysis

Definition 94 (σ algebra). *Let S be a set and let $\mathcal{F}$ be a family of subsets of $S.\mathcal{F}$ is called a σ -algebra if*

- $\emptyset, S \in \mathcal{F}$
- $(A \in \mathcal{S}) \Rightarrow (A^c = S \backslash A \in \mathcal{F})$ *(close under complements)*
- $(A_n \in \mathcal{S}, n = 1, 2, \cdots) \Rightarrow (\cup_{n=1}^{\infty} A_n \in \mathcal{F})$ *(close under countable unions / intersections)*

Definition 95. *A pair $(S, \mathcal{F})$, where S is a set and $\mathcal{F}$ is a σ -algebra of its subsets is called a measurable space. Any set $A \in \mathcal{F}$ is called an $\mathcal{F}$ -measurable set.*

Definition 96. *Given a set S and a collection $\mathcal{A}$ of subsets of S, the intersection of all σ -algebras (which is also a σ -algebras) containing $\mathcal{A}$ is called the σ -algebra generated by $\mathcal{A}$.*

Example:

- The power set of S;
- The family $\{\varnothing, S\}$ (trivial σ -algebra)
- For the set $S = \{1, 2, 3, 4\}, \mathcal{F} = \{\{1, 3\}, \{2, 4\}, \varnothing, S\}$ is also a σ -algebra
- Let $\mathcal{B}$ be the open ball in $\mathbb{R}^l, \sigma$ -algebra generated by $\mathcal{B}$ is called Borel-algebra generated by $\mathcal{B}$. Similarly, it can also be defined by open rectangles (or closed intervals, half-open intervals if in $\mathbb{R}$

Definition 97 (Measure). *Let ($S, \mathcal{F}$) be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that*

a $\mu(\emptyset) = 0$

b $\mu(A) \geq 0$ *for all* $A \in \mathcal{F}$

c μ *is countably additive: if* $\{A_n\}_{n=1}^{\infty}$ *is a countable, disjoint sequence in* $\mathcal{A}$, *then* $\mu(\cup A_n) = \sum \mu(A_n)$

Definition 98. *If furthermore* $\mu(S) < \infty$, *then* μ *is said to be a* ***finite measure*** *and if* $\mu(S) = 1$ *then* μ *is said to be a* ***probability measure****.*

Definition 99. *A triple* $(S, \mathcal{F}, \mu)$ *where* S *is a set,* $\mathcal{F}$ *is a* σ *-algebra of its subsets and* μ *is a measure on* $\mathcal{F}$ *is called a measure space. The triple is called a* ***probability space*** *if* μ *is a probability measure*

Definition 100 (Measurable Functions). *Given a measurable space* $(S, \mathcal{F})$, *a real-valued function* $f : S \to \mathbb{R}$ *is measurable with respect to* $\mathcal{F}$ *(or* $\mathcal{F}$ *-measurable) if*

$$\{s \in S \mid f(s) \leq a\} \in \mathcal{F}, \forall a \in \mathbb{R}$$

If the space is a probability space, then f *is called a random variable.*

Definition 101 (Simple Function). *Let* $(S, \mathcal{F})$ *be a measurable space, a function* $\phi : S \to \mathbb{R}$ *is called a simple function if it is of the form*

$$\phi(s) = \sum_{i=1}^{n} a_i \chi_{A_i}(s)$$

where $a_1, \cdots, a_n$ *are distinct real numbers,* $\{A_i\}$ *is a partition of* S, *and* χ_{A_i} *are indicator functions. A simple function is measurable if and only if* $A_i \in \mathcal{F}$.

Theorem 50 (Pointwise convergence preserves measurability). *Let* $(S, \mathcal{F})$ *be a measurable space, and let* $\{f_n\}$ *be a sequence of* $\mathcal{F}$ *-measurable functions converging pointwise to* f. *Then* f *is also measurable.*

Theorem 51 (Approximation of measurable functions by simple functions). *Let* $(S, \mathcal{F})$ *be measurable space. If* $f : S \to \mathbb{R}$ *is* $\mathcal{F}$ *-measurable, then there is a sequence of measurable simple functions* $\{\phi_n\}$, *such that* $\phi_n \to f$ *pointwise. If* $0 \leq f$, *then the sequence can be chosen so that*

$$0 \leq \phi_n \leq \phi_{n+1} \leq f, \forall n$$

If f *is bounded, then the sequence can be chosen so that* $\phi_n \to f$ *uniformly.*

Definition 102. *Let* $(S, \mathcal{S})$ *and* $(T, \mathcal{T})$ *be measurable spaces. Then the function* $f : S \to T$ *is measurable if the inverse image of every measurable set is measurable, i.e. if* $\{s \in \mathcal{S} : f(s) \in A\} \in \mathcal{T}$ *for all* $A \in \mathcal{T}$

Definition 103 (Measurable Selection from a Correspondence). *Let* $(S, \mathcal{S})$ *and* $(T, \mathcal{T})$ *be measurable spaces, and let* Γ *be a correspondence of* S *into* T. *Then the function* $h : S \to T$ *is a measurable selection from* Γ *if* h *is measurable and* $h(s) \in \Gamma(s), \forall s \in S$.

Theorem 52 (Measurable Selection Theorem). *Let* $S \subseteq \mathbb{R}^l$ *and* $T \subseteq \mathbb{R}^m$ *be Borel sets, with their Borel subsets* $\mathcal{S}$ *and* $\mathcal{T}$. *Let* $\Gamma : S \to T$ *be a (nonempty) compact-valued and uhc correspondence. Then there exists a measurable selection from* Γ

Some notation: $M(S,\mathcal{S})$: space of measurable, extended real-valued functions on S $M^+(S,\mathcal{S})$: space of measurable, extended real-valued, non-negative functions on S

Definition 104. *Let $\phi \in M^+(S,\mathcal{S})$ be a measurable simple function, with the standard representation $\phi(s) = \sum_{i=1}^n a_i \chi_{A_i}(s)$. Then the integral of ϕ with respect to μ is*

$$\int_S \phi(s)\mu(ds) = \sum_{i=1}^n a_i \mu(A_i)$$

Definition 105. *For $f \in M^+(S,\mathcal{S})$, the **integral of** f with respect to μ is*

$$\int_S f(s)\mu(ds) = \sup \int_S \phi(s)\mu(ds)$$

where the supremum is taken over all simple functions ϕ in $M^+(S,\mathcal{S})$ with $0 \leq \phi \leq f$. If $A \in \mathcal{S}$, then the integral of f over A with respect to μ is

$$\int_A f(s)\mu(ds) = \int_S f(s)\chi_A(s)\mu(ds)$$

Every $f \in M^+(S,\mathcal{S})$ can be written as the limit of an increasing sequence $\{\phi_n\}$ of simple functions. The next theorem tells us that the integral $\int f d\mu$ is also the unique limit t of $\int \phi_n d\mu$, i.e. it does not depend on the particular sequence $\{\phi_n\}$ chosen.

Theorem 53 (Monotone Convergence Theorem - Lebesgue). *If $\{f_n\}$ is a monotone increasing sequence of functions in $M^+(S,\mathcal{S})$ converging pointwise to f then*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Now we can think about generate the definition to functions that take negative value. Define the positive parts and negative parts as below: let $f : S \to \mathbb{R}$ be an arbitrary function. We denote

$$f^+(s) = \begin{cases} f(s) & \text{if} \quad f(s) \geq 0 \\ 0 & \text{if} \quad f(s) < 0 \end{cases}$$

and

$$f^-(s) = \begin{cases} -f(s) & \text{if} \quad f(s) \leq 0 \\ 0 & \text{if} \quad f(s) > 0 \end{cases}$$

Definition 106. *Let $(S,\mathcal{S},\mu)$ be a measure space, and let f be a measurable, real-valued function on S. If f^+ and f^- both have finite integrals with respect to μ, then f is integrable and the integral of f with respect to μ is*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Definition 107. *If $(S,\mathcal{S},\mu)$ is a probability space and f is integrable, then call $\int f d\mu$ the **expected value** of f*

9.1 Transition Functions

Definition 108 (Transition Function). *Let* $(Z, \mathcal{Z})$ *be a measurable space. A transition function is a function* $Q : Z \times \mathcal{Z} \to [0, 1]$ *such that*

- $\forall z \in Z, Q(z, \cdot)$ *is a probability measure on* $(Z, \mathcal{Z})$*, and*
- $\forall A \in \mathcal{Z}, Q(\cdot, A)$ *is a* $\mathcal{Z}$ *-measurable function.*

Interpretation: $\forall a \in Z, A \in \mathcal{Z}$

$$Q(a, A) = Pr\left\{z_{t+1} \in A \mid z_t = a\right\}$$

A Markov process can be completely described by this transition function, and the most important property of Markov process is the same transition function can be used in all periods, making each period's problem symmetric. Define T to be the operator from $M^{+}(Z, \mathcal{Z})$

$$(Tf)(z) = \int f\left(z'\right) Q\left(z, dz'\right), \forall z \in Z$$

Definition 109 (Markov operator). *Expected value of* f *next period if the current state is* z*, called the Markov operator associated with* Q*. Define* $T^{*}\lambda$ *to be the operator from the set of probability measure on* $(Z, \mathcal{Z})$:

$$\left(T^{*}\lambda\right)(A) = \int Q(z, A)\lambda(dz), \forall A \in \mathcal{Z}$$

Interpretation: probability that the state will be in A next period, given that current values of the state are drawn according to the probability measure λ.

Theorem 54. *Following holds*

- T *maps the space of bounded* $\mathcal{Z}$ *-measurable functions,* $B(Z; \mathcal{Z})$ *into itself.*
- T^{*} *maps the space of probability measures on* $(Z; \mathcal{Z})$*, that is* $\Lambda(Z, \mathcal{Z})$ *into itself*
- $$\int (Tf)(z)\lambda(dz) = \int f\left(z'\right)\left(T^{*}\lambda\right)\left(dz'\right)$$

There are other two properties a transition function may have:

Definition 110 (Feller property). *A transition function* Q *on* $(Z, \mathcal{Z})$ *has the Feller property if the associated operator* T *maps the space of bounded continuous functions on* Z *into itself; that is if* $T : C(Z) \to C(Z)$

Definition 111 (Monotone). *A transition function* Q *on* $(Z, \mathcal{Z})$ *is monotone if the associated operator* T *has the property that for every nondecreasing function* $f : Z \to \mathbb{R}$*, the function* Tf *is also nondecreasing.*

9.2 Probability Measures on Space of Sequences

Given a transition function Q on $(Z, \mathcal{Z})$, we want to look at partial (finite) histories of shocks and complete (infinite) histories generated by this transition function:

$$z^t = (z_1, \cdots, z_t), t = 1, 2, \cdots \quad z^\infty = (z_1, z_2, \cdots)$$

Let $(Z, \mathcal{Z})$ be a measurable space, and for any finite $t = 1, 2, \cdots$, let

$$\left(Z^t, \mathcal{Z}^t\right) = (Z \times \cdots \times Z, \mathcal{Z} \times \cdots \times \mathcal{Z})$$

denote the product space. We can define a measure on $(Z^t, \mathcal{Z}^t)$

$$\mu^t(z_0, \cdot) = \mathcal{Z}^t \rightarrow [0,1], \quad t = 1, 2, \ldots$$

as follow: $\forall B = A_1 \times \cdots \times A_t \in \mathcal{Z}^t$

$$\mu^t(z_0, B) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} 1 Q(z_{t-1}, dz_t) Q(z_{t-2}, dz_{t-1}) \cdots Q(z_0, dz_1)$$

This approach can also be used to define probability over infinite sequences $z^\infty = (z_1, z_2, \ldots)$ (So we will work with infinite product space $Z^\infty = Z \times Z \times \cdots$.) Define a finite measurable rectangle $B \subseteq Z^\infty$:

$$B = A_1 \times A_2 \times \cdots \times A_T \times Z \times Z \times \cdots$$

where $A_t \in \mathcal{Z}, t = 0, 1, 2, \cdots, T < \infty$. Let $\mathcal{C}$ be the family of all finite measurable rectangles, and $\mathcal{A}^\infty$ the family of all finite unions of sets in $\mathcal{C}$. Then we can show that $\mathcal{A}^\infty$ is an algebra. Let $\mathcal{Z}^\infty$ be the σ -algebra generated by $\mathcal{A}^\infty$. Define the measure similar as before;

$$\mu^\infty(z_0, B) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} Q(z_{t-1}, dz_t) Q(z_{t-2}, dz_{t-1}) \cdots Q(z_0, dz_1)$$

We can check that this measure will satisfy the three conditions imposed on measures on an algebra. By the Caratheodory and Hahn Extension Theorem, exists a unique extension of μ^∞ to $\mathcal{Z}^\infty$. Therefore, $(Z^\infty, \mathcal{Z}^\infty, \mu^\infty)$ is a probability space.

Definition 112 (Stochastic Process). *A stochastic process on* $(\Omega, \mathcal{F}, P)$ *is an increasing sequence of* σ *-algebra* $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots \subseteq \mathcal{F}$; *a measurable space* $(Z, \mathcal{Z})$; *and a sequence of functions* $\sigma : \Omega \rightarrow Z, t = 1, 2, \cdots$ *such that each* σ_t *is* $\mathcal{F}_t$ *measurable.*

Definition 113 (Stationarity). *A stochastic process is called stationary if* $P_{t+1,\cdots,t+n}$ *is independent of t, i.e*

$$F_{t_1+k, t_2+k, \cdots, t_s+k}(b_1, b_2, \cdots, b_s) = F_{t_1, t_2, \cdots, t_s}(b_1, b_2, \cdots, b_s)$$

for any finite set of indices $\{t_1, t_2, \cdots, t_s\} \subset \mathbb{Z}$ *with* $s \in \mathbb{Z}^+$, *and any* $k \in \mathbb{Z}$

Definition 114 (Markov process). *A stochastic process is called a (first-order) Markov process if*

$$P_{t+1,\cdots,t+n}(C \mid a_{t-s}, \cdots, a_{t-1}, a_t) = P_{t+1,\cdots,t+n}(C \mid a_t)$$

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