Recitations 7 - Additional meeting

Question 1 247 [I.2 Fall 2017 majors]

There are two assets: a risk-free asset with return r_f and a risky asset with return $\tilde{r}$. Returns are per-dollar returns, that is, total returns for one dollar invested. An agent has expected utility function with von Neumann (or Bernoulli) utility index $v(x) = -(\alpha - x)^2$, for $\alpha > 0$, and initial wealth $w > 0$ Assume that $\alpha > wr_f$. Negative investment (i.e., short selling) is permitted for both assets.

a Find the optimal investment in the risky asset as a function of the expected return and the variance of the return on the risky asset, the risk-free return, and the agent's wealth. Consider the following comparative statics exercise. The return $\tilde{r}$ is changed to a more risky return $\tilde{r}'$. The expected return on $\tilde{r}'$ is the same as on $\tilde{r}$, that is, $E(\tilde{r}') = E(\tilde{r})$

b State a definition of more risky return (or more risky random variable).

c Suppose that $E(\tilde{r}) > r_f$. Show that the optimal investment in the risky asset with more risky return $\vec{r}'$ is smaller then the optimal investment with less risky return $\tilde{r}$, everything else being unchanged? Prove your answer.

Question 2

There are three states with equal probabilities $\pi_s = \frac{1}{3}$ for $s \in \{1, 2, 3\}$. Consider two state contingent consumption plans $z = (8, 2, 2)$, and $y = (3, 3, 6)$

a Does y FOSD dominate z?

b Is z more risky than y?

Question 3 [Stochastic Dominance and Risk]

Consider two real-valued random variables y and z on some finite state space with $\mathbb{E}[y] = \mathbb{E}[z]$.

a Prove that if y is more risky than z, then $\mathbb{E}[v(z)] \geq \mathbb{E}[v(y)]$ for every nondecreasing continuous and concave function $v : \mathbb{R} \to \mathbb{R}$. You may assume v is twice differentiable.

b Give an example of two random variables y and z such that $y \neq z, \mathbb{E}[y] = \mathbb{E}[z]$ and neither z is more risky than y nor y is more risky than z.

Question 4 [Pratt]

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twicedifferentianble vN-M utility $v : \mathbb{R} \to \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable $\tilde{z}$ with $\mathbb{E}(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w.

a Prove that A is an increasing function of w if and only if risk compensation ρ is an increasing function of w for every $\tilde{z}$ with $\mathbb{E}(\tilde{z}) = 0$ and $\tilde{z} \neq 0$.

b Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$ where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.

c Give an example of two vN-M utility function v_1 and v_2 such that neither v_1 is more risk averse than v_2, nor v_2 is risk averse than v_1 in the sense of the Theorem of Pratt.

Question 5

Suppose that $\tilde{z}$ takes two values z_1 or z_2, where $z_1 \leq z_2$, with probabilities π and $1-\pi$, respectively. Let $\tilde{y}$ takes two values y_1 or y_2, where again $y_1 \leq y_2$, with probabilities π and $1-\pi$. Assume that $0 < \pi < 1$

a Under what conditions (necessary and sufficient) on z_i and y_i does $\tilde{z}$ dominate $\tilde{y}$ in the sense of the First-Order Stochastic Dominance? Justify your answer.

b Under what conditions does $\tilde{z}$ dominate $\tilde{y}$ in the sense of the Second-Order Stochastic Dominance? Under what conditions is $\tilde{y}$ more risky than $\tilde{z}$?

Question 6 ∼ 223 [I.2 Fall 2016 majors]

Consider two real-valued random variables y and z such that $\mathbb{E}[y] = \mathbb{E}[z]$. Suppose that cumulative distribution functions F_y and F_z have the following (weak) single-crossing property:

$$\exists_{t^* \in \mathbb{R}} \quad \begin{cases} F_y(t) \geqslant F_z(t) & \text{for } t \leqslant t^* \\ F_y(t) \leqslant F_z(t) & \text{for } t \geqslant t^* \end{cases}$$

Show that y is more risky than z. You may assume that y and z are random variables on a state space with S states.

Question 7

Consider two random variables y and z with the same expectations $\mathbb{E}[y] = \mathbb{E}[z]$.

a Show that if y is more risky than z then $\mathrm{Var}[y] \geqslant \mathrm{Var}[z]$.

b Suppose both y and z are normal. Show that y is more risky than z if and only if $\mathrm{Var}[y] \geqslant \mathrm{Var}[z]$.

c Show that if z is more risky than y and y is more risky than z then y and z have the same distribution,i.e. $F_y(t) = F_z(t)$ for all t You may assume y and z take only finitely many values.

Question 8

Consider two random variables y and z on a probability space. You may think about y and z as statecontingent consumption plans on a finite state space. Let $\{y_1, y_2, \ldots, y_k\}$ be the values y can take. Show that if $\mathbb{E}[z \mid y] = 0\,(= \mathbb{E}[z \mid y = y_i]$ for all $i \in \{1, 2, \ldots, k\})$ then $z + y$ is more risky than y

Question 9 271 [I.2 Spring 2019 majors]

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\bar{z})$ and $E(\bar{y})$ their expected values. You may assume that $\bar{y}$ and $\bar{z}$ take values in a finite interval $[a, b]$

a State a definition of $\bar{z}$ first-order stochastically dominating (FSD) $\tilde{y}$. State a definition of second-order stochastic dominance (SSD). Your definitions should be stated in terms of cumulative distribution functions F_y and F_z

b State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.

c Suppose that $\bar{z}$ has uniform distribution on an interval $[\underline{z}, \bar{z}]$ while $\bar{y}$ has uniform distribution on $[y, \bar{y}]$. Under what conditions on the bounds $\underline{z}, \bar{z}, \underline{y}, \bar{y}$ does $\bar{z}$ FSD $\tilde{y}$? Prove your statement.

d Suppose again that $\bar{z}$ and $\tilde{y}$ have uniform distributions as in (c). Show that if $\underline{y} \leq \underline{z}$ and $z \leq \bar{y}$ and $E(\bar{z}) = E(\bar{y})$, then $\tilde{y}$ is more risky than $\bar{z}$ You may assume some specific (distinct) numerical values of $\underline{z}, \bar{z}, \underline{y}, \bar{y}$ in your proof, if you find it convenient.

Question 10 255 [I.2 Spring 2018 majors]

Consider two real-valued random variables $\tilde{y}$ and $\bar{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\bar{z})$ and $E(\bar{y})$ their expected values. You may assume that $\tilde{y}$ and $\tilde{z}$ take values in a finite interval $[a, b]$

a State a definition of $\bar{z}$ second-order stochastically dominating (SSD) $\tilde{y}$. Your definition should be stated in terms of cumulative distribution functions F_y and F_z

b State a definition of $\tilde{y}$ being more risky than $\bar{z}$. Provide a justification for why it is a sensible definition of more risky.

c Extend the definition of more risky to random variables that may have different expected values as follows: $\tilde{y}$ is more risky than $\bar{z}$ if and only if $\tilde{y} - E(\tilde{y})$ is more risky than $\tilde{z} - E(\bar{z})$. Show that if $\tilde{y}$ is more risky than $\tilde{z}$ and $E(\bar{z}) \geq E(\bar{y})$, then $\bar{z}$ second-order stochastically dominates $\tilde{y}$

d Show that $\lambda\bar{z}$ is more risky than $\bar{z}$ for every $\lambda \geq 1$ and every $\bar{z}$. Note that $E(\bar{z})$ may be different from 0