Afriat's Theorem proof by Fostel, Scarf, and Todd (2004)

A version of this proof was provided by Ichiro Obara.

Definition 0.1. *We say that the observations satisfy the Generalized Axiom of Revealed Preference* ***(GARP)*** *if for every ordered subset* $\{i, j, k, \ldots r\} \subset \mathbb{N}$:

$$p_i \cdot x_j \leq p_i \cdot x_i$$

$$p_j \cdot x_k \leq p_j \cdot x_j$$

$$\vdots$$

$$p_r \cdot x_i \leq p_r \cdot x_r$$

it must be true that each inequality is, in fact, an equality

Consider a finite data set $D = \left\{(p^t, x^t) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{+}, t = 1, \ldots, T\right\}$. This note proves the following proposition, which is skipped in the class.

Theorem 0.2. *Suppose that a finite data set D satisfies GARP. Then there exists $\lambda^t > 0, U^t, t = 1, \ldots, T$ such that*

$$U^j \leq U^i + \lambda^i p^i \cdot \left(x^j - x^i\right) \text{ for all } i, j \in \{1, \ldots, T\}$$

Denote by M^T the set of $T \times T$ matrices where all diagonal elements are 0. We say that $A \in M^T$ satisfies GA if the following condition is satisfied.

$$\begin{gathered}\text{(GA) For any } \left\{a_{t(n)t(n+1)}\right\}_{n=1}^{N} \\ \text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \ldots N-1 \\ \text{then } a_{t(N)t(1)} \geq 0\end{gathered}$$

where $a_{N,N+1} = a_{N,1}$. Note that the $T \times T$ matrix where ij entry is given by $a_{ij} = p^i \cdot (x^j - x^i)$ satisfies GA if D satisfies GARP.

First we show that GA is equivalent to the following condition.

$$\begin{gathered}\text{(GA}^*\text{) For any } \left\{a_{t(n)t(n+1)}\right\}_{n=1}^{N} \\ \text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \ldots N \\ \text{then } a_{t(n)t(n+1)} = 0 \text{ for } n = 1 \ldots N\end{gathered}$$

Lemma 0.3. *GA and GA* are equivalent.*

Proof. Suppose that GA is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for all n, then any $a_{t(n)t(n+1)}$ can be regarded as the tail of a cycle (the one starting at $a_{t(n+1)t(n+2)}$) in GA. So $a_{t(n)t(n+1)} \geq 0$, hence $a_{t(n)t(n+1)} = 0$ for all n

Conversely, suppose that GA* is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for $n = 1 \ldots N-1$, then $a_{t(N)t(1)} < 0$ cannot be the case because then GA* implies $a_{t(N)t(1)} = 0$, a contradiction. Hence $a_{t(N)t(1)} \geq 0$ □

Now we prove the above proposition.

Proof. *of* ***Theorem (0.2)***.

The proof is based on induction. We show that we can find such $\lambda^t > 0, U^t, t = 1, \ldots, T$ for every $A \in M^T$ that satisfies GA* if we can find them for every $A \in M^{T-1}$ that satisfies GA*. since this is trivially true for $T = 1$ this proves that the same property holds for every T. We start with the following lemma.

Lemma 0.4. *Suppose that $A \in M^T$ satisfies GA*. Then there exists t^* such that $a_{t^*t} \geq 0$ for $t = 1, \ldots, T$*

Proof. Suppose not. Then we can construct a cycle $\left\{a_{t(n)t(n+1)}, n = 1, \ldots, N\right\}$ such that $q_{t(n)t(n+1)} < 0$ for all n. This contradicts GA*. Suppose that $A \in M^T$ satisfies GA and, without loss of generality, assume that $a_{Tt} \geq 0$ for all t. Now define $(T-1) \times (T-1)$ matrix A' as follows.

$$a'_{ij} = \begin{cases} a_{ij} \text{ if } a_{Tj} > 0 \\ \min\left\{a_{ij}, a_{iT}\right\} \text{ if } a_{Tj} = 0 \end{cases}$$

(Note that any diagonal element is zero $(a'_{ii} = a_{ii} = 0)$, so $A' \in M^{T-1}$. If not, then $a'_{ii} = a_{iT} < 0$. But $a_{iT} < 0$ and $a_{Ti} = 0$ violates GA*). □

Lemma 0.5. *If $A \in M^T$ satisfies GA*, then $A' \in M^{T-1}$ satisfies GA*.*

Proof. Suppose not. Then we can find a cycle $\left\{a'_{t(n)t(n+1)}, n = 1, \ldots, N\right\}$ such that $a'_{t(n)t(n+1)} \leq 0$ for all n such that at least one inequality is strict. If every a'_{ij} is a_{ij} in this cycle, then this contradicts the assumption that A satisfies GA *. So suppose that there exists a'_{ij} within this cycle such that $a'_{ij} = a_{iT} \leq 0$ and $a_{Tj} = 0$. Then we can replace a'_{ij} with a_{iT} and a_{Tj} in the original cycle. In this way, we can eliminate such a'_{ij} and guarantee that each element of this cycle is from A. Thus again we reach a contradiction. Hence A' must satisfy GA* □

Now we can complete the proof. By the inductive assumption, there exists $\lambda^t > 0, U^t, t = 1, \ldots, T-1$ such that

$$U^j \leq U^i + \lambda^i a'_{ij} \text{ for all } i, j \in \{1, \ldots, T-1\}$$

By definition of a'_{ij}, we have

$$U^j \leq U^i + \lambda^i a_{ij} \text{ for all } i, j \in \{1, \ldots, T-1\}$$

Define U^T and $\lambda^T > 0$ as follows.

$$U^T = \min_{i \in \{1,\dots,T-1\}} \left\{ U^i + \lambda^i a_{iT} \right\}$$
$$\lambda^T = \max \left\{ 1, \max_{j: a_{Tj} \neq 0} \left\{ \frac{U^j - U^T}{a_{TJ}} \right\} \right\}$$

We are done if we can show

$$U^T \leq U^i + \lambda^i a_{iT} \text{ for all } i$$
$$U^j \leq U^T + \lambda^T a_{Tj} \text{ for all } j$$

The first inequalities are satisfied by definition. As for the second inequality, it follows from the definition for any j when $a_{Tj} > 0$. If $a_{Tj} = 0$, then

$$U^j \leq U^i + \lambda^i a'_{ij} = U^i + \lambda^i a_{iT} \text{ for any } i \in \{1, \dots, T-1\}$$

by definition of a'_{ij}. Hence we get

$$\begin{aligned} U^j &\leq \min_{i \in \{1,\dots,T-1\}} \left\{ U^i + \lambda^i a_{iT} \right\} \\ &= U^T \\ &= U^T + \lambda^T a_{Tj} \end{aligned}$$

This proves that we can find such $\lambda^t > 0, U^t, t = 1, \dots, T$ for every $A \in M^T$ that satisfies GA*. since GA* is equivalent to GA by the lemma and GA is satisfied when D satisfies GARP, the proposition is proved. □