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Question 1 [261 IV.2 Spring 2018 majors]

Dynamic Moral Hazard

Consider a two-period principal-agent problem. In each period, there are m possible output levels for the principal, $q_i \in \mathbb{R}$, and n effort levels for the agent, $a_j \in \mathbb{R}$.
Let $\mathbb{P}(q|a)$ be the probability of q given a. The principal's per-period payoff is expected output minus payments to the agent. The agent's utility is

$$U(w,a) = v(w) - c(a)$$

where $v' > 0, v'' < 0, c' > 0, c'' > 0, v(w)$ corresponds to the agent's utility from payments by the principal and $c(a)$ corresponds to effort costs.

Time elapses as follows. First, the principal makes a take-it-or-leave-it offer to the agent, who has an outside option worth $\bar{U}$. Once everyone agrees to the contract, there are no more opportunities to quit throughout the two-period relationship.
In period 1, the agent exerts effort a_1, output q_1 realizes, and the agent is paid $w_1(q_1)$. Everyone observes q_1 at the end of period 1.
In period 2,the agent exerts effort a_2, output q_2 realizes, and the agent is paid $w_2(q_1, q_2)$.
Everyone's overall payoff is the sum of per-period payoffs.

1. Write down formally the principal's problem of
   (a) minimizing the cost of implementing a given effort profile $\mathbf{a} = (a_1, a_2)$, where $a_1 \in \mathbb{R}$ is period- 1 effort and $a_2 \in \mathbb{R}^m$ is the effort plan in period 2 contingent on output in period 1 subject to a lifetime participation constraint,
   (b) maximizing expected profit.
2. Show that, in general, w_2 will depend not only on q_2, but also on q_1.
3. Derive and interpret the inverse Euler equation from 1(a):

$$\frac{1}{v'(w_1(q_1))} = \mathbb{E}[\frac{1}{v'(w_2(q_1,q_2))}|q_1] \equiv \sum_{q_2} \frac{1}{v'(w_2(q_1,q_2))}\mathbb{P}(q_2|a_2(q_1))$$

Question 2

Consider the following principal-agent problem. Let $\Pi = \{\pi_1, \ldots, \pi_n\} \subset \mathbb{R}$ be the set of possible output levels, where $\pi_j \in \mathbb{R}$ for every j. An agent is able to exert effort at two levels, e_L and e_H. In addition, suppose that there is another random variable, or signal, y, with possible values $Y = \{y_1, \ldots, y_\ell\}$. Let

$$p(\pi, y \mid e)$$

denote the probability of output level π and signal value y when the agent's effort is e. Assume that the (marginal) distribution of output given e_H first- order stochastically dominates that given e_L, and that the agent is strictly risk averse, with preferences that are separable in effort and output.

A risk-neutral principal only cares about maximizing expected revenue net of wage costs. Assume that if the principal offers to the agent a profit-maximizing wage schedule, the agent will accept the contract and exert effort at level e_H.

a) Describe the principal's problem formally.

b) Write down the optimality conditions of this problem.

c) Show that if $p(\pi, y \mid e) = p_1(\pi \mid e)p_2(y)$ then the profit-maximizing wage schedule does not depend on y.

d) In general, any conditional probability mass function (CPMF) $p(\pi, y \mid e)$ can be reinterpreted as $p(\pi, y \mid e) = p_1(\pi \mid e)p_2(y \mid \pi, e)$ for some CPMFs p_1 and p_2. Show that if $p_2(y \mid \pi, e) = p_2(y \mid \pi)$ then the profit-maximizing wage schedule does not depend on y.

Question 3 [220 IV.1 Spring 2016 majors]

Optimal Auction

A seller owns an object, and values it at 0. There is a buyer with valuation $v \sim U[0, 1]$. The seller does not know the buyer's valuation, and designs an optimal mechanism to fulfill some objective, whereby the seller asks for the buyer's valuation and then awards the object to the buyer with probability $q(v)$ and charges the buyer an amount of money $p(v)$ if the buyer reported a valuation v.

1. Assume that the seller wants to maximize own profit, $p(v)$.

   (a) Show that the seller's virtual surplus can be written as

   $$2v - 1$$

   (b) Describe the seller's optimal auction.

2. Assume instead that the seller wants to maximize a weighted average of own profit, $p(v)$ (with weight $\alpha \in [0, 1]$), and consumer surplus, $v - p(v)$ (with weight $1 - \alpha$).

(a) Show that the seller's virtual surplus can be written as

$$(3\alpha - 1)v + 1 - 2\alpha$$

(b) Describe the seller's optimal auction as a function of $\alpha \in [0, 1]$.

Question 4 [221 IV.2 Spring 2016 majors]

The Social Value of Public Information by Morris Shin (2002 AER)
There is a continuum of agents, uniformly distributed on $[0, 1]$. Each agent $i \in [0, 1]$ chooses $a_i \in R$. Let a be the action profile. Agent i has utility function

$$u_i(a, \theta) = -\left[(1 - r)(a_i - \theta)^2 + r\left(L_i - \bar{L}\right)\right]$$

where $r \in (0, 1)$ is a constant, θ represents the state of the economy,

$$L_i = \int_0^1 (a_j - a_i)^2 \, dj \quad \text{and} \quad \bar{L} = \int_0^1 L_j dj$$

Intuitively, agent i wants to minimize the distance between his action and the true state θ, and also minimize the distance between his action and the actions of others. The parameter r represents the trade-off between these two objectives. Social welfare (normalized) is

$$W(a, \theta) = \frac{1}{1 - r} \int_0^1 u_i(a, \theta) di = -\int_0^1 (a_i - \theta)^2 \, di$$

Agent i forms expectations $E_i[\cdot] = E[\cdot|\mathcal{I}_i]$ conditional on his information $\mathcal{I}_i$ and maximizes expected utility.

1. Show that each agent i 's optimal action is given by

$$a_i = (1 - r)E_i[\theta] + rE_i[\bar{a}]$$

   where $\pi = \int_0^1 a_j dj$ is the average action. Show that if θ is common knowledge then $a_i = \theta$ for every i is an equilibrium.

2. Suppose that θ is drawn heuristically from a uniform prior over the real line. Agents observe a public signal

$$y = \theta + \eta$$

   where $\eta \sim N(0, \sigma^2)$. Therefore, $\theta | y \sim N(y, \sigma^2)$. Now, agents maximize expected utility $E[u_i|y]$ given the same public information y. Show that $a_i(y) = y$ for every i is an equilibrium. Derive the following expression for welfare given θ :

$$E[W|\theta] = -\sigma^2$$

3. Assume now that, in addition to the public signal, each agent i observes a private signal

$$x_i = \theta + \epsilon_i$$

where $\epsilon_i \sim N(0, \tau^2)$ is (heuristically) independent across i and of θ and η. Let $\alpha = 1/\sigma^2$ and $\beta = 1/\tau^2$

   a) Show that

$$E_i[\theta] = E[\theta | x_i, y] = \frac{\alpha y + \beta x_i}{\alpha + \beta}$$

   b) Suppose that there is a number κ such that for every agent j

$$a_j(x_j, y) = \kappa x_j + (1 - \kappa) y$$

   Compute the value of $E_i[\bar{a}]$ and show that

$$\kappa = \frac{\beta(1-r)}{\alpha + \beta(1-r)}$$

   defines an equilibrium.

4. Show that expected welfare is given by

$$E[W(a, \theta)|\theta] = -\frac{\alpha + \beta(1-r)^2}{[\alpha + \beta(1-r)]^2}$$

Show that

$$\frac{\partial E[W|\theta]}{\partial \beta} > 0$$

and $\frac{\partial E[W|\theta]}{\partial \alpha} \geq 0$ if and only if $\frac{\beta}{\alpha} \leq \frac{1}{(2r-1)(1-r)}$ Interpret and compare with your answer to part (b).