# Previous finals

**Question 1 [Final 2020]**

1. Prove that a finite extensive form game of perfect information has an equilibrium in pure strategies.
2. Present an example of an extensive form game that does not have an equilibrium in pure strategies. but where each player has at least one information set which is a singleton.

**Question 2 [Final 2020]**

1. Define subgame perfect equilibria and sequential equilibria.
2. State whether any of the following is true or false, and provide either a proof or a counterexample to prove your / claim:
   (a) The behavioral strategy profile of a sequential equilibrium is a subgame perfect equilibrium.
   (b) For any subgame perfect equilibrium there is a sequential equilibrium with that behavioral strategy profile

**Question 3 [Final 2020]**

In the Race $(m, k)$ game, where $m$ and $k$ are two positive integers with $m > k$, two players alternate in subtracting coins from a pile of coins with $m$ coins initially. Player 1 starts, and can take away 1 or 2 or $, \dots k$ coins (note he must take away at least 1). Then the move goes to player 2 who can do the same from the remaining pile. The player who removes the last coin wins.

1. Write the extensive form game for Race$(5, 3)$.
2. Write the set of pure strategies for player 1 in Race$(5, 3)$.
3. Find the optimal strategy profiles (that is, find out (1) when a player can win and (2) how he can win when he can) for the Race (5,3) game.
4. Find the optimal strategy profiles for the Race $(m, k)$ game.

**Question 4 [Final 2020]**

The Backward Induction procedure is based on replacing the nodes that have all immediate successors in the set of final nodes, in finite games that are also of perfect information. We consider here whether one can extend this procedure the other games

1. Consider first the case in which some of the information sets are not singletons
   a) Can you identify a "last information set" (that is an information set such that all immediate successors of all the nodes in the set are in the set of final nodes?
   b) When such a set exists, what can you do?
2. Consider next the case in which the game has countably many nodes.
   a) Can you define a backward induction procedure?
   b) Can you modify the definition in some way to implement the idea of backward induction in this case?

**Question 1 [Final 2019]**

1. Prove that for any finite EFG of perfect information, there is a last move node, that is a move node $x$ such that $IS(x) \subseteq Z$.
2. Prove, or disprove by showing a counter-example to the statement: In any finite EFG of perfect recall, there is a last information set $I^i$ for some player $i$, that is, an information set such that for any node $x \in I^i, IS(x) \subseteq Z$.

**Question 2 [Final 2019]**

In this question we will consider only finite Extensive Form Games (EFG) of perfect recall. Take a behavioral strategy profile $\hat{\beta}$ of the EFG. A one-stage deviation from $\hat{\beta}^i$ for player $i$ is a behavioral strategy $\gamma^i$ of $i$ that differes from $\hat{\beta}^i$ at a single information set.

1. Define EFG of perfect information.
2. The One Stage Deviation Principle for finite EFG of perfect information is the following statement:

   "A behavioral strategy profile is a subgame perfect equilibrium of the EFG of perfect information if and only if for no player and no subgame there is a one-stage deviation in the subgame that gives to the deviating player a strictly larger payoff in that subgame, than the original strategy."

   Prove the One Stage Deviation Principle for games of perfect information.

**Question 3 [Final 2019]**

Consider again the One Stage Deviation Principle.

1. State the principle for games for general extensive form games with perfect recall, but not necessarily of perfect information. Do you need some additional condition in your statement, as compared to the state of the principle in the case of games of perfect information?
2. Prove your claim.

**Question 4 [Final 2019]**

1. The Three players matching pennies is, in word, the following extensive form game.

   a) Player 1 moves first, and chooses head $(H^1)$ or tails $(T^1)$. This choice is not communicated to anyone else.
   b) After the choice of player 1 , player 2 moves second, and chooses heads $(H^2)$ or tails $(T^2)$.This choice is not communicated to anyone else.
   c) Player 3 is told whether the choices of player 1 and 2 matched (that is, they were either $H^1$ and $H^2$, or $T^1$ and $T^2$) , or they did not, but not the specific choices. Below, "match" and "not match" will be defined analogously.
   d) Player 3 then moves, and chooses heads $(H^3)$ or tails $(T^3)$ ; then the game ends.
   e) Payoffs are as follow:

   - If the choice of player 3 matches that of player 1, then player 1 pays two units each to player 2 and 3, so payoffs are (-4,2,2)
   - If the choice of player 3 does not match that of player 1, then player 2 and 3 pay two units each to player 1, so payoffs are (4,-2,-2)

   Please address the following questions:

   (a) Write the extensive form of the game;
   (b) Find the Nash equilibria of the game;
   (c) Find the sequential equilibria of the game.

2. Now consider again the Three players matching pennies above, but now allow players 1 and 2 to exit at their information set, in the order. Payoffs at these additional end notes are as follows:

   - If player 1 exits, the game ends and payoffs are (-2,1,1);
   - If player 2 exits, the game ends and payoffs are (2,-1,-1); The payoffs at other end nodes are as in the previous game. Please address the following questions

   (a) Write the extensive form of the game
   (b) Find the Nash equilibria of the game
   (c) Find the sequential equilibria of the game.

**Question 1 [Final 2018]**

Consider the following game:

- $I = \{1, 2\}$. Nature moves, and decides a quantity $M$ which can take values 2 and 5 with probability $p \in [0, 1]$.$M$ is told to the player 1 but not told to the player 2 .
- Player 1 has to decide how much out of 10 units he transfers to the player 2. This amount, denoted by $a$, must be an integer. $a$ is multiplied by $M$ and the total amount $Ma$ is then communicated to player 2.
- Player 2 can then decide an amount $b$, non-negative integer less or equal than $Ma$ to be transferred back to player 1.
- Final payoffs are $(10 - a + b, Ma - b)$. Then the game ends.

a) Write it as a EFG.

b) Find the Nash equilibria of this game.

c) Find the subgame perfect equilibria for this game.

**Question 2 [Final 2018]**

In the following, fix the set of players $I$ and for each player $i \in I$ fix the set of actions $A^i$.

1. The Nash equilibrium existence theorem shows that one can associate a set of mixed strategy profiles to every profile of utilities $\left(u^i\right)_{i\in I}$. This defines the Nash equilibrium correspondence. Prove that this correspondence is u.h.c. Define carefully all objects you define.
2. Characterize the most general form of transformations on utilities that leaves the best response unchanged.

**Question 3 [Final 2018]**

Consider a game: Find all the sets that survive IEWDS.

| | L | C | R |
|---|---|---|---|
| T | (1,2) | (2,3) | (0,3) |
| M | (2,2) | (2,1) | (3,2) |
| B | (2,1) | (0,0) | (1,0) |

**Question 4 [Final 2018]**

1. Present an examples of an EFG that does not have equilibrium in pure strategies but where each player has at least one information set which is a singleton.
2. In a finite EFG let $Y$ be non-empty subset of the set of nodes of the tree $X$, with the partial order induced by the restriction of the order of the tree. Prove that there is an element in $Y$ with no immediate successors.
3. Prove that in finite EFG there is a node $x$ which has all the immediate successors in the set of final nodes.
4. Prove that in an EFG of perfect information every node defines a subgame.

**Question 1 [Final 2017]**

A normal form game has $I$ players, action sets $A^i$ and utility functions $u^i$, for $i \in I$. The vector $u^i : i = 1, \ldots, n$ is a utility profile. Fix the set of players and the action sets. The Nash equilibrium correspondence (NEC) associates to every utility profile the set of Nash equilibria of the normal form game with that profile.

1. Prove that one can restrict the domain of the correspondence to a compact set of ntility profiles.

2. Prove that the NEC is closed valued.

3. Prove that the NEC is upper-herni-continuous, or find an example to show it is not

**Question 2 [Final 2017]**

Define a Perfect Equilibrium of a normal form game as the limit of sequences of Nash equilibria of a perturbed game.

1. Prove that a perfect equilibrium is a Nash equilibrium.

2. A strategy is fully mixed if it gives positive probability to all the actions in the action set of the player, and a strategy profile is fully mixed if the strategies of all players are fully mixed. Prove that a fully mixed strategy profile is perfect.

3. Give as simple sufficient condition for a Nash equilibrium to be perfect.

**Question 3 [Final 2017]**

As in question 1, fix the set of players and the action sets.

1. Identify the set of all transformations of the utility function of a player that leave the Best Response correspondence unchanged. Please note you will state and prove statement like "The set of transformations $T$ (put the definition here) of the utility functions of player 1 is such that for all normal formal games with that player set and action sets the Best Response for the utility function is $u^1$ and for its transformation $T\left(u^1\right)$ are the same."

2. Prove your answer in detail.

**Question 4 [Final 2017]**

1. Consider the utility function of player 1 :

| | L | C | R |
|---|---|---|---|
| T | 7 | 5 | 1 |
| M | 1 | 4 | 3 |
| B | 4 | 1 | 7 |

   What are the correlated strategies for which player 1 will want to follow the recommended action for each of his actions?

2. What are the set of correlated equilibria of a zero-sum game?